Random Walks

October 24, 2024

1 Random walks in one dimension

1.1 Arithmetic sequences

Problem 1.1

Alice buys a Pistachio farm for 1,000 dollars. Each month, she harvests and sells 40 dollars worth of pistachios. After 12 months, how much money has she made selling pistachios? Is she better off, i.e. has she regained all the money she spent?

Problem 1.2

Let's assume Alice the Pistachio farmer has no money, and she borrows the 1,000 dollars from the bank to buy her pistachio farm. Let a_n denote the amount of money she has after farming pistachios for n months. Find a formula for a_n.

Solution. $a_n = -1,000 + n \cdot 40$. □

Definition 1.3

An arithmetic sequence a_n is a (possibly infinite) collection of numbers with constant difference, i.e.

$$a_n - a_{n-1} = \text{ a constant.}$$

Problem 1.4

Let

$$f_n, \quad n = 0, 1, \cdots, N$$

be a collection of numbers satisfying

$$f_n = \frac{f_{n+1} + f_{n-1}}{2}, \quad n = 1, \cdots, N-1.$$

Prove that the maximum of the numbers f_n is achieved when $n = 0$ or N,

$$\max_{n=0,1,\cdots,N} f_n = \max\{f_0, f_N\}.$$

Solution. f_n is equal to it's local average so it has no interior maximum or is the constant sequence. □

Problem 1.5

Let

$$f_n, \quad n = 0, 1, \cdots, N$$

be a collection of numbers satisfying

$$f_n = \frac{f_{n+1} + f_{n-1}}{2}, \quad n = 1, \cdots, N-1.$$

Show that f_n is an arithmetic sequence.

Solution. Modify f_n by subtracting $an + b$ from f_n so that $b = f_0$ and $aN + b = f_N$. Now note that the modified f_n still satisfies the local average property, and the endpoints are 0. Hence, it must be constant 0.

You can also prove this by rearranging the equality and working backwards to f_0. □

Problem 1.6 (Harder)

How many sequences of numbers $(f_0, f_1 \cdots, f_N)$ are there so that

$$f_n = \frac{f_{n+1} + f_{n-1}}{2}, \quad n = 1, \cdots, N-1$$

and $f_0 \in \{1, \cdots, 10\}$ and $f_N \in \{4, \cdots, 11\}$?

Solution. By uniqueness of arithmetic sequences with given boundary data it is just $10 * 8 = 80$. □

1.2 A Fair Casino

A casino offers the following game: you can bet 1 dollar on a coin flip, if the coin comes up heads you win 1 dollars, if the coin comes up tails you lose 1 dollars. The casino allows you to play as many times as you want.

Problem 1.7

Suppose first that the coin is fair, do you expect to be better or worse off after playing the game 100 times? If you made 10 dollars after 30 rounds, how much money would you expect to have after the next 30 rounds?

Solution. You don't expect to make any money. □

Problem 1.8

Bob shows up to the casino. He has 14 dollars. He noticed a silk suit in the casino lobby which costs $12,000$ dollars.

Bob decides he will play the casino's game until he gets $12,000$ dollars or goes broke. What is the probability that Bob will get the suit?

Hint: think about arithmetic sequences.

Solution. The probability that Bob will go broke starting at d dollars is an arithmetic sequence in d with endpoint values 1 and 0.

So it should be like $\frac{14}{12{,}000}$. □

Problem 1.9

Bob did not get the suit. He has come back the next day and decides to play the casino's game just for fun.

Bob enters the casino with 0 dollars and will borrow money whenever he doesn't have enough. After playing the game for n rounds, what is the probability that Bob has exactly 0 dollars?

What is the probability Bob has 0 dollars after n rounds if he entered with 100 dollars?

Solution. If n is odd, parity says the probability is 0.

If $n = 2m$ is even, then there must be exactly m heads. This can happen in $\binom{2m}{m}$ ways each with probability 2^{-2m}.

A similar analysis works if you start with 100 dollars, you need 100 more tails than heads. If there are enough rounds to accomodate this, a similar binomial formula will work. □

Problem 1.10 (Harder)

Bob plays the game 10,000 times during the night and starts with 10 dollars. Show that

$$\text{the probability Bob had more than 500 dollars at some point during the night} \\ = \\ 2 \times (\text{the probability that Bob has 500 dollars when he leaves the casino}).$$

Hint: draw a picture of the amount of money in Bob's wallet as the night goes on.

Solution. Consider any night when Bob makes more than 500 dollars. Freeze the point when he hits 500 dollars. At that point, he has an equal chance of making and losing money in his future playing. Therefore, $\frac{1}{2}$ of the future outcomes will end up with him having more than 500 dollars giving the desired equality. ☐

1.3 *Geometric sequences*

Problem 1.11

Ingrid the investor buys a bespoke collateralized debt obligation (BCDO) for 1,000 dollars. Every year, the value of the BCDO decreases by 7% of its previous amount. In 12 years, how much money has Ingrid gained or lost?

Solution. Compute

$$1,000 \cdot 0.93^{12}.$$

☐

Definition 1.12

An geometric sequence b_n is (possibly infinite) collection of nonzero numbers with constant ratio, i.e.

$$\frac{b_n}{b_{n-1}} = \text{ a constant.}$$

Problem 1.13

Which of the following are geometric sequences?

1. $6, 12, 24, \cdots,$
2. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots,$

3. $2, 3, 5, 7, 9, 11, \cdots$,

4. $1, 1, 1, 1, \cdots$.

Solution. All except the primes. □

Problem 1.14

Let r be a number not equal to 1. Show that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

Hint: multiply $1 + r + \cdots + r^n$ by r.

1.4 A rigged casino

The owners of the casino realize that they can make more money if the game is rigged. They offer players the same game but this time the coin is biased. *In the rigged version of the game* the coin shows up heads with probability p between 0 and 1.

Problem 1.15

Bob returns to the casino a year later. The coin is now heads with proabability p.

Bob has 100 dollars. He still wants to buy the silk suit. However, the suit has gone up in price and now costs 15,000 dollars. Bob decides to play the casino's game until he can buy the suit or goes broke. What is the probability that Bob gets the suit?

Hint: solve some of the following problems first.

Solution. Call the probability that Bob gets the suit starting with n dollars b_n. Note that b_n satisfies,

$$b_n = pb_{n+1} + (1 - p)\, b_{n-1}.$$

By the next problem, the sequence $b_n - b_{n-1}$ is a geometric sequence with common ratio

$$\frac{b_{n+1} - b_n}{b_n - b_{n-1}} = \frac{p}{1 - p}.$$

And we know $b_N = 1$ and $b_0 = 0$.

So,

$$b_2 - b_1 = \frac{p}{1 - p} b_1$$

and more generally,

$$b_{i+1} - b_i = \left(\frac{p}{1 - p}\right)^i b_1.$$

Hence, summing up

$$b_{i+1} - b_1 = b_{i+1} - b_i + b_i - b_{i-1} + \cdots + b_2 - b_1 = \sum_{k=1}^{i} \left(\frac{p}{1-p} \right)^k b_1.$$

And hence,

$$b_{i+1} = \sum_{k=0}^{i} \left(\frac{p}{1-p} \right)^k b_1$$

Now, using the geometric series formula is equal to

$$b_1 \frac{1 - \left(\frac{p}{1-p} \right)^{i+1}}{1 - \frac{p}{1-p}}$$

if $p \neq \frac{1}{2}$.

Now, choosing $i = N - 1$ and using that $b_N = 1$ we get

$$1 = b_1 \frac{1 - \left(\frac{p}{1-p} \right)^{i+1}}{1 - \frac{p}{1-p}}$$

which gives that

$$b_1 = \frac{1 - \frac{p}{1-p}}{1 - \left(\frac{p}{1-p} \right)^{i+1}}.$$

Finally, we can use the formula for b_i to get that

$$b_i = b_1 \frac{1 - \left(\frac{p}{1-p} \right)^{i+1}}{1 - \frac{p}{1-p}}$$

and substitute in b_1. □

Problem 1.16

Let

$$b_n, \quad n = 0, \cdots, N$$

be a sequence of numbers so that

$$b_n = p b_{n+1} + (1-p) b_{n-1}, \quad n = 1, \cdots, N-1.$$

Suppose that $0 < p < 1$. Is b_n a geometric sequence? Is $b_n + b_{n-1}$ a geometric sequence? Is $b_n - b_{n-1}$ a geometric sequence?

Solution. It turns out only $b_n - b_{n-1}$ is a geometric sequence. □

Problem 1.17

How many geometric series $(c_0, \cdots, c_{300})$ are there so that $c_0, c_{300} \in \{1, \cdots, 5\}$?

Solution. By uniqueness of geometric sequences, there are 5^2 such sequences. □

Problem 1.18

Bob believes the casino may have made a mistake. He believes there is a chance he can become "infinitely rich" if he plays the casino game forever. He has entered the casino with 10 dollars. For what values of p (probability of heads) is there a positive chance that Bob can play forever and never go broke?

Solution. Using the solution of problem 1.15 we have that if $p > 0.5$ then

$$\lim_{N \infty \infty} b_i = 1 - \left(\frac{p}{1-p}\right)^i > 0$$

for all i and if $p \leq 0.5$ then

$$\lim_{N \to \infty} b_i = 0.$$

□

1.5 *Introducing the expected value*

Every time Bob plays the game at the Casino, the outcome is random: it depends on the coin flip. There are many quantities that we can measure as Bob's night progresses:

- The total amount of money Bob has in his wallet after he finishes playing,
- The maximum amount of money Bob had at any point during the night,
- The absolute value of how much money Bob has won or lost at the end of the night,
- ...

All of these quantities are random (they depend on the outcome of coin flips) and are therefore called *random variables*.

Definition 1.19

Any number Z which depends on the outcome of a sequence of coin flips is called a *random variable*.

Example 1.20

If we flip n coins, and define

$$Z_i = +1 \text{ if the } i\text{th flip is heads and } -1 \text{ otherwise.}$$

Then, Z_i is a random variable.

Problem 1.21

We are back at the casino. Bob entered with 0 dollars and will play n times. He will borrow money whenever he needs to keep playing. Let Z_i be the random variable defined in the previous example. Let S_n be the random variable which represents how much money Bob has after playing n times. Prove that

$$S_n \overset{\text{def}}{=} Z_1 + \cdots + Z_n.$$

Solution. This is basically the definition of the random walk. □

Even though the outcome of a random variable is . . . well . . . random, we can still make a good guess as to what it's value will be. The best guess for a random variable is called the *expectation*.

Definition 1.22

If a random variable Z takes on finitely many values in a set S. Then we define the *expectation* of Z to be

$$\mathbb{E}\left[\, Z\right] \overset{\text{def}}{=} \sum_{s \in S} s \cdot (\text{probability that } Z = s).$$

Problem 1.23

Prove the *linearity of expectation*. If Z, W are random variables which take on finitely many values in $\mathbb{Z}$, prove that

$$\mathbb{E}\left[\, Z + W\right] = \mathbb{E}\left[\, Z\right] + \mathbb{E}\left[\, W\right].$$

Solution. Using this definition, we have to use that

$$\begin{aligned}
\mathbb{E}\left[\, Z + W\right] &= \sum_{n \in \mathbb{Z}} n \cdot \mathbb{P}\left\{Z + W = n\right\}, \\
&= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (k + (n - k)) \cdot \mathbb{P}\left\{Z = k, W = n - k\right\}, \\
&= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} k \mathbb{P}\left\{Z = k, W = n - k\right\} + \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (n - k)\, \mathbb{P}\left\{Z = k, W = n - k\right\}, \\
&= \mathbb{E}\left[\, Z\right] + \mathbb{E}\left[\, Y\right].
\end{aligned}$$

□

Problem 1.24

Bob is playing at the Casino with a *fair coin*. He wants to have a good guess about how much money he expects to lose *or* gain during this evening in absolute value.

To do this, he wants to compute the expectation of the following random variable

$$(\text{amount of money Bob has won or lost after playing } n \text{ times})^2 .$$

Using the notation of Problem 1.21, compute $\mathbb{E}\left[\, S_n^2 \right]$.

Hint: Use 1.21 and the linearity of expectation.

Solution. Write $S_n = Z_1 + \cdots + Z_n$ then

$$\mathbb{E}\left[\, S_n^2 \right] = \sum_{i,j} \mathbb{E}\left[\, Z_i Z_j \right] = n.$$

since the off diagonal terms are 0. □

Problem 1.25 (Harder, feel free to skip)

Bob wants to compute the expected absolute value of how much he will win or lose after playing n rounds of the game. In other words, Bob wants to compute the expectation of the following random variable:

$$|\text{amount of money Bob has won or lost after playing } n \text{ times}|.$$

Using the notation of Problem 1.21, compute $\mathbb{E}\left[\, |S_n| \right]$. Once again assume Bob is playing with a fair coin.

Hint: work directly from the definition of expected value.

Solution. This is actually nontrivial. The only thing I can think of is to compute for each m,

$$\mathbb{P}\left\{ |S_n| = m \right\}.$$

Which is equal to

$$2\mathbb{P}\left\{ \#\text{ heads } - \#\text{ tails } = m \right\}.$$

Since $h + t = n$ and $h - t = m$ we can solve this as

$$\mathbb{P}\left\{ 2h = m + n \right\} = \binom{n}{(n+m)/2} \frac{1}{2^n}.$$

However, this will not give an exact formula.

□

Problem 1.26

Bob is having fun. He has decided to spend the rest of his life at the Casino betting on this game. He starts with 0 dollars, and will borrow money whenever necessary to keep playing.

Let V_n be the random variable which counts how many times Bob has 0 dollars in his wallet after n coin flips. Prove that

$$\mathbb{E}[V_n] = \sum_{k=0}^{n} (\text{probability that } S_k = 0)$$

where S_k is the random variable that tells us how much money Bob has after playing k times.

1. Compute the probability that $S_k = 0$. (It depends on k!).
2. Assume for this problem that $n!$ is roughly equal to $\sqrt{n}\,(n/e)^n$ where e is a constant roughly equal to $2.718\cdots$. Prove that there is a number γ so that the probability from part 1. is roughly equal to n^{γ}.
3. Perhaps, with the help of the instructor, use this to show that for any large number M, there is n sufficiently large that
$$\mathbb{E}[V_n] \geq M.$$

If V is the random variable that counts the number of times Bob's wallet will return to having 0 dollars, then this problem tells us that

$$\mathbb{E}[V] = \infty.$$

Solution.

1. It is $\binom{2m}{m}2^{-2m}$ if $k = 2m$ is even and 0 otherwise.
2. The answer is $\frac{1}{\sqrt{n}} = n^{-\frac{1}{2}}$
3. The sum of $n^{-\frac{1}{2}}$ diverges.

□

Problem 1.27

Bob starts with 0 dollars and intends to play forever. Let q denote the probability that Bob's wallet will return to 0 at some point in the future. Using the preceding problem, prove that $q = 1$.

Solution. 1. Suppose that the probability is $0 < q < 1$.

Then we can give the distribution for $V = \lim_n V_n$, the variable which counts the total number of visits to 0.

First,

$$\mathbb{P}\{V = 1\} = (1 - q)$$

since this is the probability of never returning.

$$\mathbb{P}\{V = k\} = q^k (k - 1).$$

Hence,

$$\mathbb{E}[V] = \sum_{k=1}^{\infty} k\mathbb{P}\{V = k\} = \sum_{k=1}^{\infty} kq^k (k - 1) = \frac{1}{1 - q} < \infty$$

which is a contradiction.

2. Note that

$$\mathbb{E}[V] = 1 + q\mathbb{E}[V]$$

by definition of V and q.

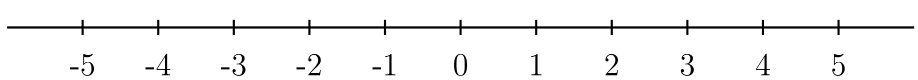
Indeed, we visit 0 once for sure because we start at 0.

We visit 0 again with probability q. After that, we expect to visit 0 a further $\mathbb{E}[V]$ many times which gives the desired equality.

□

2 Random walks on Graphs

Our analysis of Bob has secretely been an analysis of a random walk on the integers. The integers are a graph which looks like the following picture:



Bob has been walking on the integers in a random fashion, Bob's position on the graph represents how much money he currently has.

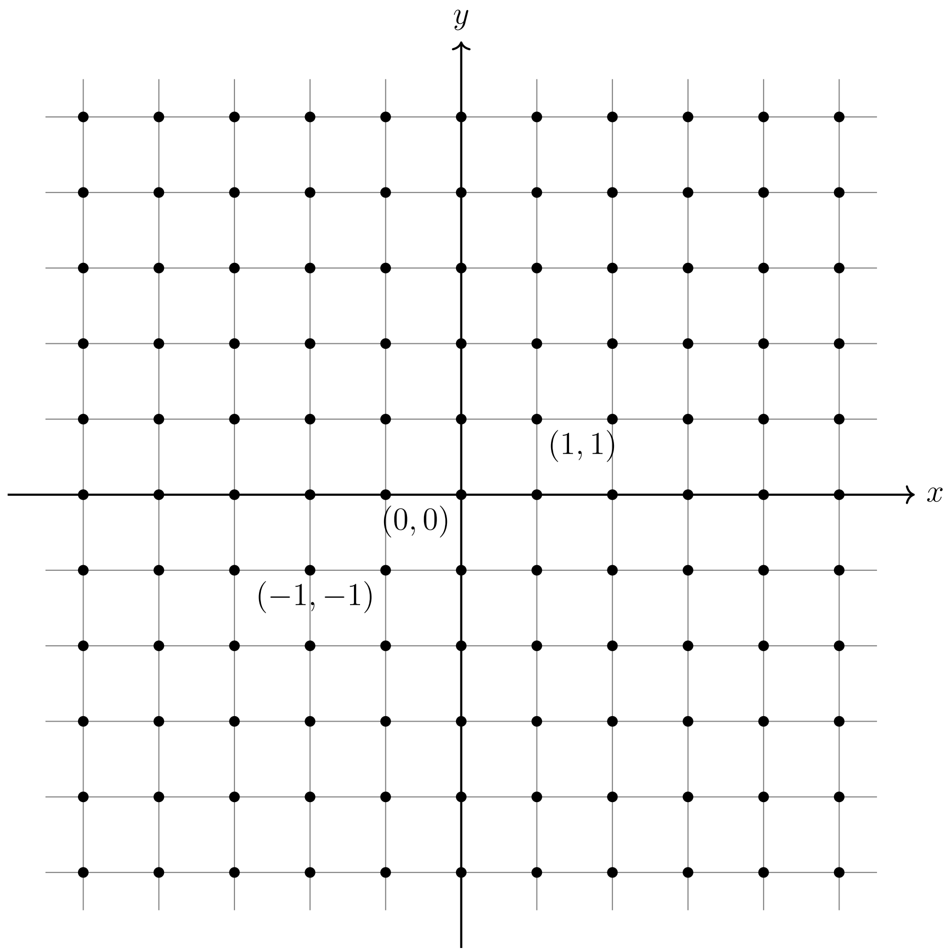
More generally, we can do a random walk on the integer lattice $\mathbb{Z}^d$.

Definition 2.1

The integer lattice $\mathbb{Z}^d$ is defined as the set

$$\mathbb{Z}^d = \{\mathbf{z} = (z_1, \cdots, z_n) \mid z_i \in \mathbb{Z}\}.$$

The picture shows the integer lattice $\mathbb{Z}^2$.



Definition 2.2

For a point $\mathbf{z} \in \mathbb{Z}^d$, the neighbors of $\mathbf{z}$ are the points

$$(z_1 \pm 1, z_2, \cdots, z_n), (z_1, z_2 \pm 1, \cdots, z_n), \cdots, (z_1, z_2, \cdots, z_n \pm 1).$$

Problem 2.3

On the picture of the lattice $\mathbb{Z}^2$, highlight the neighbors of the point $(1, 1)$.

Problem 2.4

How many neighbors does $\mathbf{z} \in \mathbb{Z}^d$ have?

Definition 2.5

A *random walk* on $\mathbb{Z}^d$ is a walk where at each step, the walker moves to a neighbor chosen at random with equal probability.

2.1 Recurrence

Problem 2.6

A random walker in $\mathbb{Z}^2$ takes left/right steps or up/down steps. Suppose a random walker walks for $2n$ steps, prove that the probability that

$$\text{\# left/right steps} \quad \text{and \# up/down steps} \quad \text{are both even}$$

is equal to $1/2$.

Solution. If either quantity is even then the other is even and by symmetry this probability will be $\frac{1}{2}$. □

Problem 2.7 (Expected distance in $\mathbb{Z}^d$)

A random walker in $\mathbb{Z}^d$ wants to keep track of his distance to the origin $\mathbf{0} \in \mathbb{Z}^d$.

He defines a distance as follows: two positions $\mathbf{z}, \mathbf{w} \in \mathbb{Z}^d$ have the following distance,

$$\|\mathbf{z} - \mathbf{w}\| \stackrel{\text{def}}{=} \sqrt{(z_1 - w_1)^2 + \cdots + (z_d - w_d)^2}$$

where $\mathbf{z} = (z_1, \cdots, z_d)$.

1. Let $\mathbf{1} \in \mathbb{Z}^d$ be the point with coordinates

   $$\mathbf{1} \stackrel{\text{def}}{=} (1, 1, \cdots, 1)$$

   what is the distance between $\mathbf{1}$ and $\mathbf{0}$.

2. The random walker has noticed the following interesting thing: if he defines the "generalized angle" between two positions $\mathbf{z}, \mathbf{w} \in \mathbb{Z}^d$ as

   $$\langle \mathbf{z}, \mathbf{w} \rangle = z_1 w_1 + \cdots + z_d w_d$$

then $\|\mathbf{z}\|^2 = \langle \mathbf{z}, \mathbf{z} \rangle$. Prove this.

3. Prove that if $\mathbf{z}, \mathbf{w}, \mathbf{v} \in \mathbb{Z}^d$ are three positions, then

$$\langle \mathbf{z} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{z}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

where the position $\mathbf{z} + \mathbf{v} \in \mathbb{Z}^d$ is defined to be

$$\mathbf{z} + \mathbf{w} \stackrel{\text{def}}{=} (z_1 + w_1, \cdots, z_d + w_d) \,.$$

4. The random walker wants a good estimate for the random quantity $\|S_n\|^2$ where $S_n \in \mathbb{Z}^d$ represents his position after taking n steps starting at 0. Compute

$$\mathbb{E}\left[\, \|S_n\|^2 \right].$$

Hint: try to express $S_n = X_1 + \cdots + X_n$ where $X_i \in \mathbb{Z}^d$ represents each step the random walker took and the addition happens coordinate wise. Then use the fact that $\|S_n\|^2 = \langle S_n, S_n \rangle$.

Solution. The answer is $\mathbb{E}\left[\, \|S_n\|^2 \right] = n^{-d}$ □

Problem 2.8

In the previous question we computed $r := \sqrt{\mathbb{E}\left[\, \|S_n\|^2 \right]}$. This represents the "expected distance" from the origin. For this question, assume the following two facts are true:

1. With probability equal to 1, the random walker's position after n steps satisfies $\|S_n\| \leq 10r$.

2. The random walker's position after n steps will be a random position, and all the points $\mathbf{z}$ so that $\|\mathbf{z}\| \leq 10r$ are equally likely to be where the random walker finishes their walk.

Compute the probability that the random walker is at 0 after n steps.

Solution. $n^{-\frac{d}{2}}$. □

Problem 2.9

Let V_n be the random variable which counts the number of times the random walker has visited 0 after taking n steps. Prove that

$$\mathbb{E}\left[\, V_n \right] = \sum_{k=0}^{n} (\text{probability that } S_k = 0)$$

where S_k denotes the position of the random walker at $k = 0$.

Solution. Proof is linearity of expectaion. □

Problem 2.10

Prove that if $d = 2$ then for any $M \geq 0$ there is n so that

$$\mathbb{E}\left[V_n\right] \geq M.$$

On the other hand, show that if $d \geq 3$ then there is a finite number $C > 0$ so that

$$\mathbb{E}\left[V_n\right] \leq C$$

for all n.

Solution. $n^{-\frac{d}{2}}$ is summable for $d \geq 3$ but not for $d = 2$. □

Problem 2.11

Prove that a random walker on $\mathbb{Z}^2$ who starts at 0 returns to 0 infinitely often.

On the other hand, prove that a random walker on $\mathbb{Z}^d$ for $d \geq 3$ may not return to 0 with positive probability.

Solution. The solution is the same as in the one dimensional case. □

2.2 *Boundary value problems in higher dimensions*

Let $A \subset \mathbb{Z}^d$ be a *finite* subset.

Definition 2.12

Define the *outer boundary of A in $\mathbb{Z}^d$* by

$$\partial A \stackrel{\text{def}}{=} \left\{\mathbf{z} \in \mathbb{Z}^d \setminus A \mid \mathbf{z} \text{ is the neighbor of a point in } A\right\}.$$

Definition 2.13

Define the *closure of A* to be

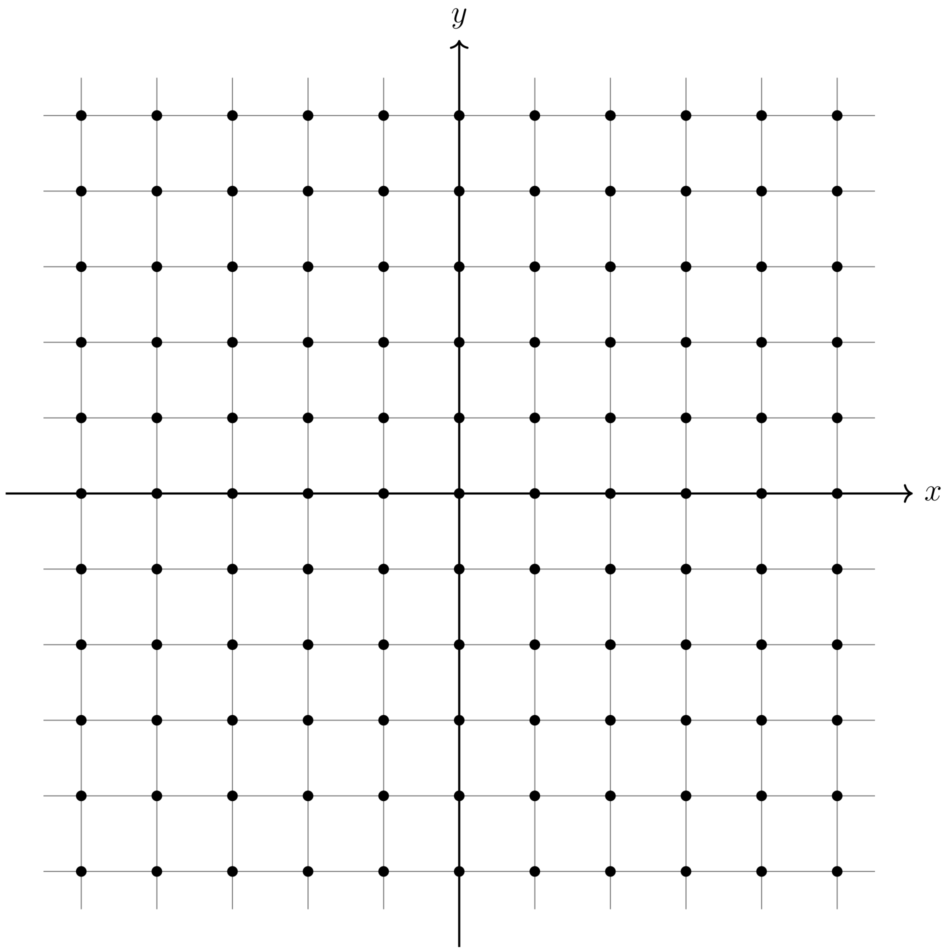
$$\bar{A} \stackrel{\text{def}}{=} A \cup \partial A.$$

Problem 2.14

Let $A \subset \mathbb{Z}^2$ be defined as

$$\{(z_1, z_2) \mid |z_1| \leq 4 \text{ and } |z_2| \leq 2\} \cup \{(3,3)\}\,.$$

On the picture draw A. What is ∂A? What is $\bar{A}$?



Definition 2.15

Let $F : \mathbb{Z}^d \to \mathbb{R}$ be a function. Define the *discrete laplacian of* F to be

$$\mathcal{L}F\left(\mathbf{z}\right) = \frac{1}{2d} \sum_{\mathbf{w} \text{ is a neighbor of } \mathbf{z}} \left(F\left(\mathbf{w}\right) - F\left(\mathbf{z}\right)\right)$$

Problem 2.16

Describe in words what $\mathcal{L}$ does to a function F.

Solution. At each point x, $\mathcal{L}F(x)$ is the difference between the value $F(x)$ and the average of F at its neighboring points. □

Problem 2.17

Let $A \subset Z^d$ be a finite subset. Suppose that $F : \bar{A} \to \mathbb{R}$ is a function which satisfies

$$\mathcal{L}F(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in A.$$

Show that

$$\max_{\mathbf{z} \in \bar{A}} F(\mathbf{z}) = \max_{\mathbf{z} \in \partial A} F(\mathbf{z}).$$

Solution. Since F is equal everywhere to it's local averages, F has no interior maximum. □

Problem 2.18

Let $A \subset Z^d$ be a finite subset. Suppose that $F_1 : \bar{A} \to \mathbb{R}$ and $F_2 : \bar{A} \to \mathbb{R}$ are two functions which satisfy

$$\mathcal{L}F_1(\mathbf{z}) = 0 = \mathcal{L}F_2(\mathbf{z}) \text{ for all } \mathbf{z} \in A.$$

Suppose moreover that $F_1(\mathbf{z}) = F_2(\mathbf{z})$ for all $\mathbf{z} \in \partial A$. Show that $F_1 = F_2$.

Solution. Consider $F_1 - F_2$, this function has no interior maximum or minimum and is zero on the boundary therefore $F_1 = F_2$. □

Problem 2.19

Let $A \subset Z^d$ be a finite subset and let $g : \partial A \to \mathbb{R}$ is a function.
Let $S_n^{\mathbf{z}}$ be a random walker on $\mathbb{Z}^d$ starting at $\mathbf{z} \in \bar{A}$. Let T be the first time the random walker hits ∂A. Define

$$F(\mathbf{z}) = \mathbb{E}\left[g\left(S_T^{\mathbf{z}}\right)\right].$$

Show that

$$\mathcal{L}F(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in A$$

and

$$F(\mathbf{z}) = g(\mathbf{z}) \text{ for all } \mathbf{z} \in \partial A.$$

Solution. It is clear that $F(x) = g(x)$ on the boundary ∂A.

On the other hand a random walker visits each of its neighbors with equal probability and therefore $\mathcal{L}F(x) = 0$. □