Powers and Logarithms I **ORMC Intermediate 2**

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Warm-up

Problem 1 *Once Mary ate a half of the peaches from a can, the level of the syrup decreased by one third. How much would decrease the syrup level from the new one, if Mary eats a half of the peaches left?*

Powers and Logarithms

Let $b > 0$. The following are the fundamental laws of exponential functions.

$$b^x \times b^y = b^{x+y} \tag{1}$$

$$b^0 = 1 \tag{2}$$

$$b^{-x} = \frac{1}{b^x} \tag{3}$$

$$\frac{b^x}{b^y} = b^{x-y} \tag{4}$$

$$(b^x)^y = b^{xy} \tag{5}$$

$$\sqrt[n]{b} = b^{\frac{1}{n}} \tag{6}$$

$$\sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m = b^{\frac{m}{n}} \tag{7}$$

Problem 2 *Use the laws of the exponents to represent the following as powers of two.*

- $2^5 \times 8^2 =$

- $\dfrac{2^5 \times 4^3}{16^2} =$

Problem 3 *Use the laws of the exponents to simplify the following as much as you can.*

- $(a^x)^2 \times a^y \times a^{-2x} =$

- $\dfrac{(a^{-x})^{-y} \times a^3}{a^{xy-3}} =$

Problem 4 *Derive property 2 of exponential functions from property 1.*

Problem 5 *Derive property 3 from properties 1 and 2.*

Problem 6 *Derive property 4 from properties 1 and 3.*

Problem 7 *Derive property 6 from property 5.*

Problem 8 *Derive property 7 from properties 5 and 6.*

Problem 9 *To what power should one raise 10 to get 10,000,000?*

Problem 10 *To what power should one raise 100 to get 10,000,000?*

Problem 11 *To what power should one raise 0.5 to get 256?*

Problem 12 *To what power should one raise* 1/9 *to get 3?*

Problem 13 *Solve the following equation.*

$$3^{12x} = \left(\frac{1}{9}\right)^3$$

The function $\log_b x$, called *logarithm to base b of x*, answers the following question. *To what power should I raise the base b to get x?* For example, $\log_2 8 = 3$.

Problem 14 *Find the following logarithms.*

- $\log_2 1024 =$

- $\log_{\frac{1}{2}} 1024 =$

- $\log_{10} 1000 =$

- $\log_{10} 0.001 =$

Problem 15 *Use the definition of the logarithm to prove that $b^{\log_b x} = x$ and $\log_b b^y = y$.*

Properties of logarithms

Since logarithmic functions are inverses of the exponential ones, properties of the logarithms mirror the formulas 1 - 5.

$b^x \times b^y = b^{x+y}$	$\log_b(pq) = \log_b p + \log_b q$
$b^0 = 1$	$\log_b 1 = 0$
$b^{-x} = \dfrac{1}{b^x}$	$\log_b \frac{1}{p} = -\log_b p$
$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b \left(\dfrac{p}{q}\right) = \log_b p - \log_b q$
$(b^x)^y = b^{xy}$	$\log_b p^y = y \log_b p$

Problem 16 *Use 1 to prove the following formula.*

$$\log_b(pq) = \log_b p + \log_b q \tag{8}$$

Problem 17 *Use 2 to prove the following formula.*

$$\log_b 1 = 0. \tag{9}$$

Problem 18 *Use 3 to prove the following formula.*

$$\log_b \left(\frac{1}{p}\right) = -\log_b p \tag{10}$$

Problem 19 *Prove the following formula.*

$$\log_b \left(\frac{p}{q}\right) = \log_b p - \log_b q \tag{11}$$

Problem 20 *Use 5 to prove the following formula.*

$$\log_b p^y = y \log_b p \tag{12}$$

Problem 21 *How is* $\log_b \sqrt{x}$ *related to* $\log_b x$*?*

Problem 22 *How is* $\log_b \sqrt[n]{x}$ *related to* $\log_b x$*?*

Problem 23 *Find the following values.*

- $\log_{\sqrt{3}} 81 =$

- $\log_6 3 + \log_6 2 =$

- $\log_2 2\sqrt{2} =$

- $2 \log_4 2 =$

Problem 24 *Solve the following equation.*

$$4^{\sqrt{x+1}} = 64 \times 2^{\sqrt{x+1}}$$

Carbon Dating

^{14}C is a radioactive isotope of carbon with *half-life time* $t_{1/2} = 5,700$ years. This means that if the amount of ^{14}C you have right now is $A_0 = 10$ grams, then 5,700 years later the amount left will be a half of that, $A(5,700) = 5$ grams.

Problem 25

- *What amount of the isotope would be left* $2 \times 5,700 = 11,400$ *years from now?*

- $3 \times 5,700 = 17,100$ *years from now?*

- $4 \times 5,700 = 22,800$ *years from now?*

- $n \times 5,700$ *years from now?*

Problem 26 *If the original amount of ^{14}C was A_0, how much would be left T years from now? The following formula should help.*

$$T = \frac{T}{t_{1/2}} \times t_{1/2} \tag{13}$$

^{14}C is produced by cosmic rays bombarding nitrogen atoms in the Earth's atmosphere. The resulting radioactive carbon combines with atmospheric oxygen to form radioactive carbon dioxide gas CO_2. The latter is digested by plants in the process of photosynthesis. It is passed further down the food chain from plants to plant-eating animals to carnivores. A living plant, or animal, has a stable level of ^{14}C in its body. Once the plant, or animal, dies, it stops exchanging the isotope with the environment. The level of ^{14}C starts decreasing according to the equation you have found in Problem 26. The idea to use the process of the ^{14}C decay for dating biological samples belongs to Willard Libby who received the Nobel Prize for his work in 1960.

Problem 27 *A petrified redwood log has 1% of the original* ^{14}C *amount. How old is the sample?*