Los Angeles Math Circle

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Euler's number

The goal of this mini-course is to give an accurate construction of the Euler's number e, one of the most fundamental constants in mathematics, physics, economics, and finance. The construction and the applications of the number e to finance and probability are broken into steps and presented as series of problems for students to solve. Harder problems are marked with the red pepper sign.

Math Circle Instructors: problems marked by the Ⓢ sign have solutions at the end of the booklet. Preparing to teach this class, try to solve them yourselves. This will deepen your understanding of the course. Only use our solutions if stuck. Do not print out the solutions for students.

1 Completeness of real numbers

The following statement is known as the *Dedekind completeness axiom*.

Axiom 1 *Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ such that $a \leq b$ for any $a \in A$ and $b \in B$. Then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for any $a \in A$ and $b \in B$.*

Intuitively, the axiom states that the set of real numbers has no holes. This is not the case for the set of rational numbers $\mathbb{Q}$. Although there exists a rational number in between any two rational numbers, the set of rational numbers has holes in the following sense.

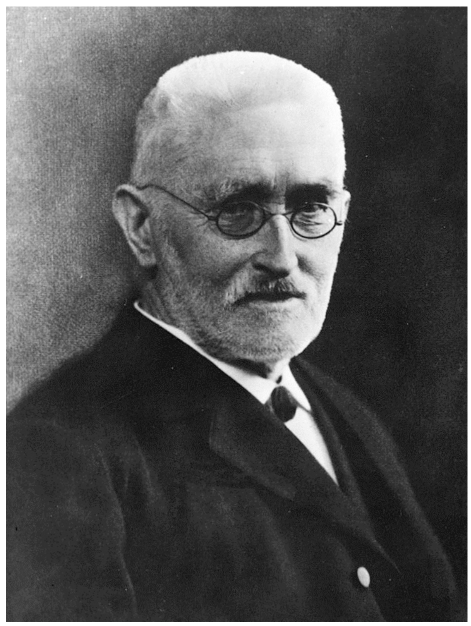
Let $A = \{a \in \mathbb{Q} : 0 \leq a \text{ and } a^2 \leq 2\}$ and $B = \{b \in \mathbb{Q} : 0 \leq b \text{ and } b^2 \geq 2\}$. Then $a \leq b$ for any $a \in A$ and $b \in B$. However, there is no such rational number c that $a \leq c \leq b$ for any $a \in A$ and $b \in B$ because $\sqrt{2} \notin \mathbb{Q}$.

Problem 1 *Prove that $\mathbb{Q}$ is countable.*

Problem 2 *Prove that $\mathbb{R}$ is not countable.*

In view of problems 1 and 2, the set of rational numbers has holes everywhere!

Axiom 1 is named after a great German mathematician Julius Wilhelm Richard Dedekind (1831 - 1916), the first man to give an accurate description of real numbers.



Richard Dedekind

Let $A \subset \mathbb{R}$. The number $u \in \mathbb{R}$ is called an *upper bound* of A, if $a \leq u$ for any $a \in A$. A set A having an upper bound is called *bounded from above.*

A real number s is called the *supremum of A* a.k.a. the *least upper bound of A* if

- s is an upper bound of A and
- for any upper bound u of A, $s \leq u$.

To shorten notations, they write $s = \sup A$.

Problem 3 *Let* $s = \sup A$ *for* $A \subset \mathbb{R}$*. Prove that* s *is unique.*

Problem 4 $A = \{(a_n)_{n=1}^{\infty} : a_n = 0.\underbrace{33\ldots3}_{n\ times}\}$*. Show that* $\sup A = 1/3$.

Problem 5 *Let* $A = \{a \in \mathbb{R} : 0 \leq a$ *and* $a^2 \leq 2\}$*. Prove that* $\sup A = \sqrt{2}$.

Lemma 1 *A subset of real numbers bounded from above has the least upper bound.*

Problem 6 *Prove lemma 1.*

A sequence of real numbers $(a_n)_{n=1}^{\infty}$ is called *monotonically increasing* if $m < n \Rightarrow a_m < a_n$.

Problem 7 *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence of real numbers and let* A *be a real number. Use the* $\epsilon - N$ *formalism to explain the meaning of the following mathematical sentence.*

$$\lim_{n\to\infty} a_n = A$$

Lemma 2 *Let* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *be sequences of real numbers. Let* $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$*. Then the following is true.*

- $\lim_{n\to\infty} (a_n + b_n) = A + B$
- $\lim_{n\to\infty} (a_n - b_n) = A - B$
- $\lim_{n\to\infty} a_n b_n = AB$

  If $B \neq 0$*, then also*

- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$

Problem 8 *Prove lemma 2.*

The following lemma will be crucial for defining the number e in the next section.

Lemma 3 *A monotonically increasing sequence of real numbers bounded from above has a limit.*

Problem 9 Ⓢ *Prove lemma 3.*

2 Defining e

Problem 10 *Recall and prove the binomial identity.*

Problem 11 Ⓢ *Prove that*

$$\left(1+\frac{1}{n}\right)^n < 3 - \frac{1}{n} \tag{1}$$

for $n = 3, 4, \dots$

Problem 11 shows that the sequence

$$e_n = \left(1+\frac{1}{n}\right)^n, \; n = 1, 2, \dots \tag{2}$$

is bounded from above, $e_n < 3$ for $n \in \mathbb{N}$.

The following very useful statement is known as *Bernoulli inequality.*

$$(1+x)^n \geq 1 + nx \text{ for } x \geq -1 \text{ and } n \in \mathbb{N} \tag{3}$$

Problem 12 *Use induction to prove 3.*

Problem 13 Ⓢ *Use Bernoulli inequality to prove that the sequence e_n defined by (2) is monotonically increasing.*

Problems 11, 13 and lemma 3 show that the sequence $(e_n)_{n=1}^{\infty}$ has a limit.

$$e \stackrel{\text{def}}{=} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{4}$$

Problem 14 *Prove the following formula.*

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = e \tag{5}$$

Problem 15 *Prove the following formula.*

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = e \tag{6}$$

Problem 16 Ⓢ *Prove the following formula.*

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n=\frac{1}{e} \tag{7}$$

Problem 17 *Let $x \in \mathbb{R}$. Prove the following formula.*

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=e \tag{8}$$

Hint: if $x > 0$, then $\lfloor x \rfloor \le x \le \lceil x \rceil$.

Problem 18 *Prove the following formula.*

$$\lim_{t\to 0}(1+t)^{\frac{1}{t}}=e$$

Problem 19 *Prove the following formula.*

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x \tag{9}$$

Note that (7) is a particular case of (9) for $x = -1$.

The following very important formula will be proven in a Calculus course.

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=2+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots \tag{10}$$

Problem 20 *Find the first six significant digits of e.*

3 Compounded interest

Let P be the primary capital invested at a constant rate r compounded annually. Let $V(t)$ be the value of the investment in t years.

Problem 21 *Derive the formula for $V(t)$.*

Problem 22 *Derive the formula for $V(t)$ if the annual rate r is compounded monthly.*

Problem 23 *Derive the formula for $V(t)$ if the annual rate r is compounded n times a year, $n \in \mathbb{N}$.*

Problem 24 *Derive the formula for $V(t)$ if the annual rate r is compounded continuously.*

Problem 25 *Which of the investments described in problems 21, 22, and 24 is a better choice? Why?*

4 Euler's number and probability

Problem 26 *A gambler plays a slot machine n times. Each time, his chance to win is p. What is his chance to win k times?*

Problem 27 *A gambler plays 10,000 times a slot machine that pays out one time in 10,000. What is the chance that the gambler loses every bet?*

5 Solutions

Problem 9 Since $(a_n)_{n=1}^{\infty}$ is bounded from above, it has the least upper bound. Let $a = \sup\{a_n\}$. Then $a_n \le a$ for $n \in \mathbb{N}$.

Let us take a real number $\epsilon > 0$. If $\{a_n\}_{n=1}^{\infty} \cap (a-\epsilon, a] = \emptyset$, then $a_n < a - \epsilon/2$ for $n \in \mathbb{N}$. This contradicts the fact that a is the least upper bound. The contradiction shows that $\{a_n\}_{n=1}^{\infty} \cap (a-\epsilon, a] \neq \emptyset$.

Let N be the first natural number such that $a_N \in (a-\epsilon, a]$. Since the sequence is monotonically increasing, $a_n \in (a-\epsilon, a]$ for all $n \ge N$. Thus, $\lim_{n\to\infty} a_n = a$.

Problem 11

$$\left(1+\frac{1}{n}\right)^n < 2 + \sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^k} = 2 + \sum_{k=2}^{n} \frac{1}{k!} \frac{n(n-1)\cdot \ldots \cdot (n-k+1)}{n^k} <$$

$$< 2 + \sum_{k=2}^{n} \frac{1}{k!} < 2 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 2 + 1 - \frac{1}{n} = 3 - \frac{1}{n}$$

Problem 13 The inequality

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$$

is equivalent to the following one.

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}} > \left(1+\frac{1}{n}\right)^{-1} = 1 - \frac{1}{n+1}$$

It's not hard to check that

$$\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}} = 1 - \frac{1}{(n+1)^2}.$$

Thanks to Bernoulli inequality,

$$\left(1-\frac{1}{(n+1)^2}\right)^{n+1} > 1-\frac{n+1}{(n+1)^2} = 1-\frac{1}{n+1} \quad Q.E.D.$$

Problem 16 $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{(-1)(-n)} = \lim_{n\to\infty}\left(\frac{n}{n-1}\right)^{-n} = \lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^{n(-1)} = \lim_{n\to\infty}\left(\left(1+\frac{1}{n-1}\right)^{(n-1)\frac{n}{n-1}}\right)^{-1} = \frac{1}{e}.$