ORMC Intermediate 2 **October 20, 2024**

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Reviewing Recursion

Recall that *recursion* is a method of solving problems that contain smaller versions of themselves as subproblems. A recursive function will make calls to itself in order to solve these sorts of problems.

Problem 1 *The Fibonacci sequence is a sequence of numbers whose first few terms are*

$$1, 1, 2, 3, 5, 8, 13...$$

In general, the n^{th} *term of the sequence is defined by*

$$F_n = F_{n-1} + F_{n-2}.$$

Use Python to write a recursive function $fib(n)$ *that returns the* n^{th} *term in the Fibonacci sequence (try not to use any loops inside of your function!).*

Problem 2 *Given two nonnegative integers* $0 \leq b \leq a$*, the binomial coefficient* $\binom{a}{b}$ *is the number of ways of choosing* b *objects from a collection of* a *objects, disregarding order. Its name* *comes from the fact that it can be computed as the coefficient of* x^b *in the expansion of* $(1+x)^a$. *The binomial coefficients satisfy the recursive relation*

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b},$$

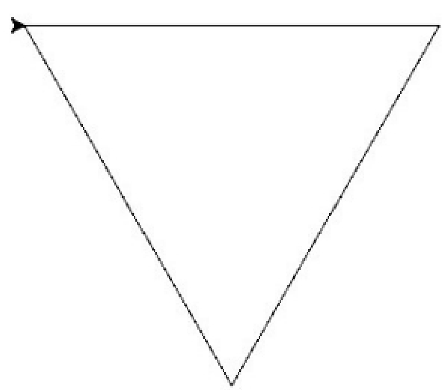
as well as the boundary values

$$\binom{a}{0} = \binom{a}{a} = 1.$$

Use Python to write a recursive function $binom(a, b)$ *computing the binomial coefficient* $\binom{a}{b}$.

Figuring out the area of Koch snowflake

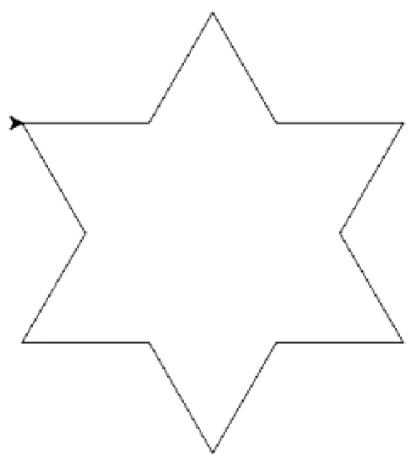
The following equilateral triangle KS_0 of side length a (drawn using the Python's Turtle) is the base step for constructing a beautiful closed curve known as the *Koch snowflake*.



During the 10/13 class, we have found the area of the triangle.

$$A_0 = \frac{\sqrt{3}}{4} a^2 \tag{1}$$

The figure KS_1 is the next step of the construction.



The area A_1 of KS_1 is the sum of A_0, the area of the base triangle, and of the areas of the three new spikes, each of them an equilateral triangle of side length $a/3$. To find the area of a spike, we can use formula 1 with a replaced by $a/3$, the side length of the spike.

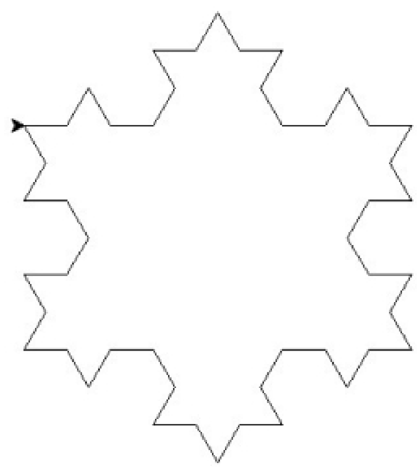
$$\text{Spike area} = \frac{\sqrt{3}}{4} \left(\frac{a}{3}\right)^2 = \frac{\sqrt{3}}{4} \frac{a^2}{9} = \frac{A_0}{9}$$

Therefore, $A_1 = A_0 + 3 \times \frac{A_0}{9} = A_0 + \frac{A_0}{3}$.

$$A_1 = A_0 \left(1 + \frac{1}{3}\right) \tag{2}$$

For the reason that will become clear soon, we will not simplify formula 2 any further.

The figure KS_2 below is the next step.



The area A_2 of KS_2 is the sum of A_1 and the areas of the 12 spikes KS_1 sprouts. A spike is an equilateral triangle of side length $a/9$. To compute the area of the spike, we can use formula 1 with a replaced by $a/9$.

$$\text{Spike area} = \frac{\sqrt{3}}{4}\left(\frac{a}{9}\right)^2 = \frac{A_0}{81}$$

Therefore,

$$A_2 = A_0\left(1 + \frac{1}{3} + \frac{12}{81}\right) = A_0\left(1 + \frac{1}{3} + \frac{4}{27}\right)$$

$$A_2 = A_0\left(1 + \frac{1}{3} + \frac{1}{3} \times \frac{4}{9}\right) \tag{3}$$

To trace the changes, let us start forming the following table.

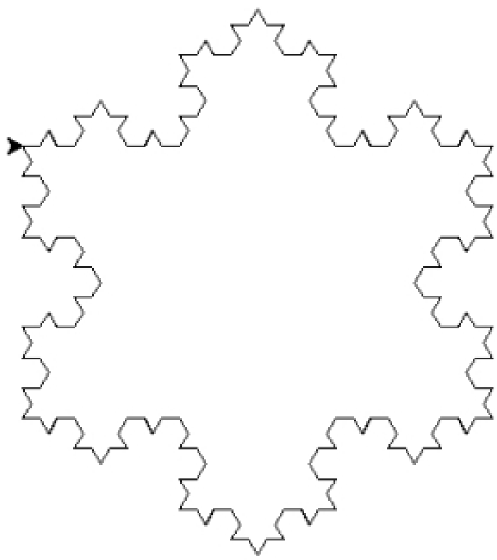
n	0	1	2
side length	a	$\frac{a}{3}$	$\frac{a}{9}$
number of sides	3	12	48
number of new spikes	0	3	12

At the next step of the construction, each of the 48 sides of KS_2 sprouts a spike, an equilateral triangle of side length $a/27$. To compute the area of the spike, we can use formula 1 with a replaced by $a/27$.

$$\text{Spike area} = \frac{\sqrt{3}}{4}\left(\frac{a}{27}\right)^2 = \frac{A_0}{27^2}$$

Therefore,

$$A_3 = A_2 + \frac{48}{27^2}A_0.$$



$$A_3 = A_0 \left(1 + \frac{1}{3} + \frac{1}{3} \times \frac{4}{9} + \frac{1}{3} \times \left(\frac{4}{9} \right)^2 \right) \tag{4}$$

Note that at each step of the construction, the number of sides quadruples. We remove the middle of each side, breaking the former into two, and add two more sides to fill the gap. At the next step, each of the sides sprouts a triangle, adding to the area. Let us check this observation by taking a look at the extension of our table.

n	0	1	2	3	4	5
side length	a	$\frac{a}{3}$	$\frac{a}{3^2}$	$\frac{a}{3^3}$		
number of sides	3	3×4	3×4^2	3×4^3		
number of new spikes	0	3	3×4	3×4^2		

Problem 3 *Fill the table for* $n = 4, 5$.

Problem 4 *Check if the following formula*

$$A_4 = A_0 \left(1 + \frac{1}{3} + \frac{1}{3} \times \frac{4}{9} + \frac{1}{3} \times \left(\frac{4}{9} \right)^2 + \frac{1}{3} \times \left(\frac{4}{9} \right)^3 \right) \quad (5)$$

correctly represents the area of KS_4.

Problem 5 *Guess the pattern and write down the formula for A_5. Generalize to any $n = 1, 2, 3, \ldots$*

$A_5 =$

$A_n =$

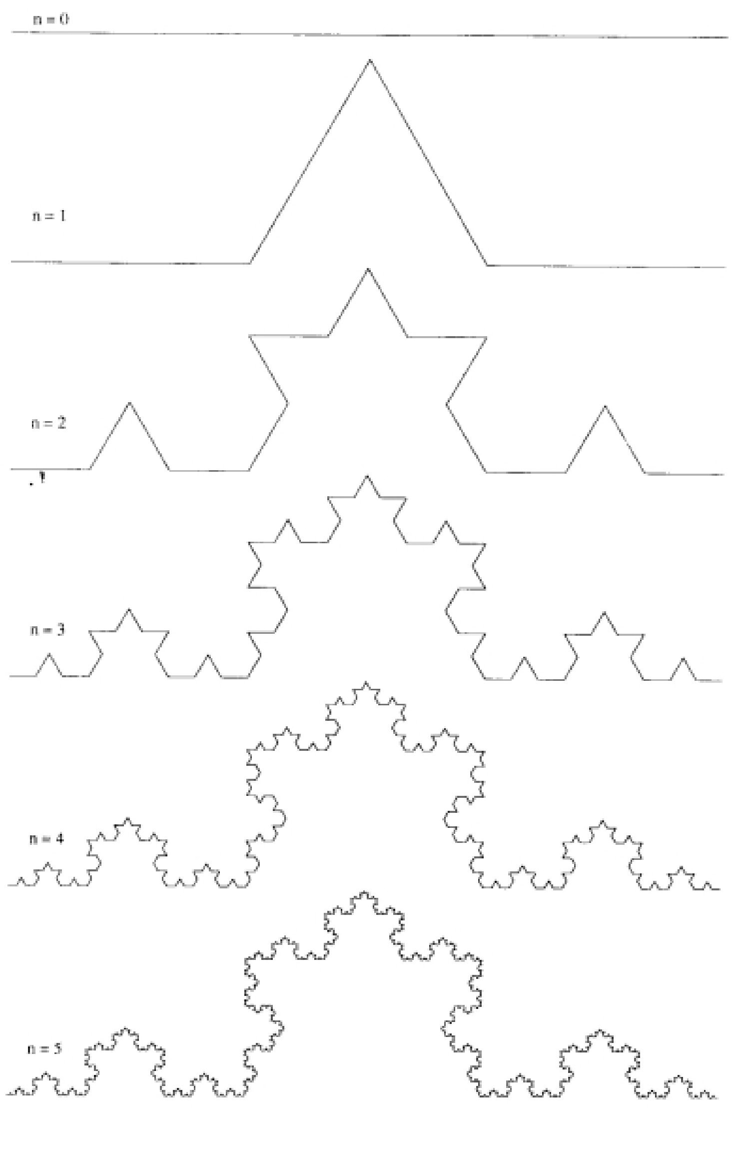
Let $S = \sum_{n=1}^{\infty} \frac{1}{3} \times \left(\frac{4}{9}\right)^{n-1} = \frac{1}{3} + \frac{1}{3} \times \frac{4}{9} + \frac{1}{3} \times \left(\frac{4}{9}\right)^2 + \frac{1}{3} \times \left(\frac{4}{9}\right)^3 + \ldots$

Problem 6 *Find S. Hint: find $4/9 \times S$ and compare it to S.*

Problem 7 *Find the area A of the Koch snowflake* $KS = \lim_{n\to\infty} KS_n$.

Please get back to Problem 11 of the previous handout:

Problem 8 *In the Turtle module, write a function kochs(n,a) that draws a Koch snowflake of order n and base length a.*
Hint: It might be helpful to use the function kochc(n,a) you defined to draw Koch curves as depicted below.



If you are finished doing all of the above but there still remains some time... give a try to the "Tower of Hanoi" puzzle at

https://www.mathsisfun.com/games/towerofhanoi.html.

The puzzle with n disks naturally admits a recursive solution. See if you can figure it out!