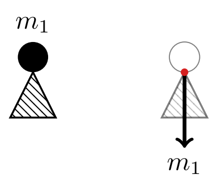
Geometry of Masses I

Prepared by Sunny & Mark on October 11, 2024

Instructor's Handout

Part 1: Balance

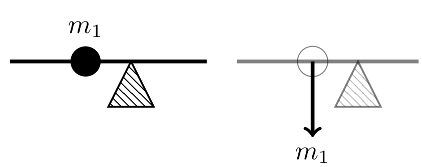
Example 1:
Consider a mass m_1 on top of a pin in two-dimensional space.
Due to gravity, the mass exerts a force on the pin at the point of contact.
For simplicity, we'll say that the magnitude of this force is equal the mass of the object— that is, m_1.



The pin exerts an opposing force on the mass at the same point, and the system thus stays still.

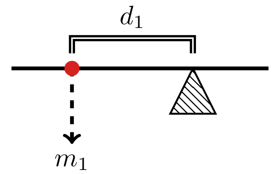
Remark 2:
Forces, distances, and torques in this handout will be provided in arbitrary (though consistent) units.
We have no need for physical units in this handout.

Example 3:
Now attach this mass to a massless rod and try to balance the resulting system.
As you might expect, it is not stable: the rod pivots and falls down.



This is because the force m_1 is offset from the pivot (i.e, the tip of the pin).
It therefore exerts a *torque* on the mass-rod system, causing it to rotate and fall.

Definition 4: Torque
Consider a rod on a single pivot point. If a force with magnitude m_1 is applied at an offset d from the pivot point, the system experiences a *torque* with magnitude $m_1 \times d$.

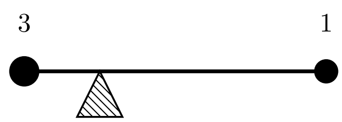


We'll say that a *positive torque* results in *clockwise* rotation, and a *negative torque* results in a *counterclockwise rotation*. As stated in Remark 2, torque is given in arbitrary "torque units" consistent with our units of distance and force.

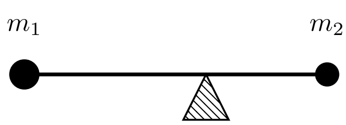
Look at the diagram above and convince yourself that this convention makes sense:
- m_1 is positive (masses are usually positive)
- d_1 is negative (m_1 is *behind* the pivot)
- therefore, $m_1 \times d_1$ is negative.

Definition 5: Center of mass
The *center of mass* of a physical system[1] the point at which one can place a pivot so that the total torque the system experiences is 0.
In other words, it is the point at which the system may be balanced on a pin.

Problem 6:
Consider the following physical system: we have a massless rod of length 1, with a mass of size 3 at position 0 and a mass of size 1 at position 1. Find the position of this system's center of mass.



Problem 7:
Do the same for the following system, where m_1 and m_2 are arbitrary masses.



Solution

The CoM will be such that the rod is split into $d_1 + d_2 = 1$ according to the relation $m_1 d_1 = m_2 d_2$. This is sufficient, but if you want to solve for one of the d, you get $d_1 = \frac{m_2}{m_1+m_2}$. This should be intuitive; the distance of each mass from the CoM is proportional to the other mass's share of the total mass.

[1] a *physical system* is just an arrangement of things toward which we apply certain laws of physics

Problem 8:
Consider n masses $m_1, m_2, ..., m_n$. Place each m_i at position x_i.
Find the resulting system's center of mass.
(Ask an instructor if you get stuck, because you need this for the rest of the packet.)

Solution

$x_0 = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$ where $M = \sum_{i=1}^{n} m_i$

Problem 9:
Extend Problem 8 into two dimensions:
Place n masses $m_1, m_2, ..., m_n$ at positions $(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3)$ on a massless plane.
Find the coordinates of the resulting system's center of mass.
Hint: If a plane balances on a pin, it does not tilt in the x or y direction.

Solution

$x_0 = \sum_{i=1}^{n} m_i x_i / \sum_{i=1}^{n} m_i \qquad y_0 = \sum_{i=1}^{n} m_i y_i / \sum_{i=1}^{n} m_i$

Part 2: Continuous mass

Now let's extend this idea to a *continuous distribution* of masses rather than discrete point masses. This isn't so different; a continuous distribution of mass is really just a lot of point-masses, only that there are so many of them so close together that you can't even count them[2]. In general, finding the CoM requires integral calculus, but not always...[3]

Problem 10:
You are given a cardboard cutout of a seahorse and some office supplies. How might you determine the CoM? Does your strategy also work in 3D?

Solution

Many correct answers. One example:
(1) Stick a thumb tack into the horse and let it come to equilibrium
(2) Use a ruler or string to draw a straight line through that point along the direction of gravity
(3) Repeat (1) and (2) at a different point
(4) The intersection of the two lines marks the CoM

Problem 11:
Recall the system from Problem 7. Suppose we replace point masses m_1 and m_2 with cardboard seahorses, also with mass m_1 and m_2. We make sure to place the centers of mass of each seahorse exactly where the point masses used to be. Must we now move the pivot to keep the system balanced? Why or why not?

Solution

Objects can be treated as point masses located at their CoM without affecting the system.

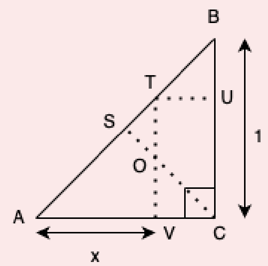
[2]For example, your pencil might seem like a continuous distribution of mass, but it's really just a whole lot of atoms.
[3]Many of the following problems can be solved with integration even though you're meant to solve them without it. But remember, in math, whenever you accomplish the same task two different ways, that really means that they're somehow the same thing.

Definition 12: Centroid
Centroids are closely related to, and often synonymous with, centers of mass. A centroid is the geometric center of an object, regardless of the mass distribution. Thus, the centroid and center of mass are the same when the mass is uniformly distributed.

Problem 13:
Where is the centroid of a right isosceles triangle? How does this extend to all right triangles? How does it extend to all isosceles triangles?

Solution

There are probably some other clever ways of doing this without calculus but here's one way:



Clearly, the centroid O must be somewhere along SC. Now we just need to find s in terms of x, that is, the balancing point along either of the shorter sides. To do this, we split the triangle into three regions: $\triangle AVT$ on the left, the rectangle $TUCV$ on the right, and $\triangle TUB$ also on the right.
Each region exerts a torque proportional to its area times the horizontal distance from VT of that region's centroid. Note that, even though we don't know what x is yet, we can use it to find itself. By similar triangles, the centroids of $\triangle AVT$ and $\triangle TUB$ are both located $x(1-x)$ away from VT. The centroid of $TUVC$ is trivially $\frac{1-x}{2}$. So we get the following equation:

$$\frac{1}{2}x^2 \cdot x(1-x) = x(1-x) \cdot \frac{1-x}{2} + \frac{1}{2}(1-x)^2 \cdot x(1-x) \quad .$$

We easily find that $x = \frac{1}{3}$. Remarkably, the ratio $\frac{SO}{SC}$ is also $\frac{1}{3}$.
Any right triangle is just an isosceles right triangle that's been scaled along some axis, so the centroid scales with it and this one-third rule still applies.
Any isosceles triangle is just two right triangles, so *its* centroid will be in between its two "sub-centroids" from each right triangle, that is, one-third the altitude.

Problem 14:
(Bonus) How might you find the centroid of any triangle? Why does this work?

Solution

It turns out that all three medians of a triangle always intersect at a single point. That point is the centroid. You could feasibly guess this by taking what you learned from Problem 13 and applying Cavalieri's Theorem. Otherwise, I'm interested to see what students come up with.

center of mass is something you encounter every day and that you are already very familiar with, whether you know it or not. Otherwise, you'd be falling over all the time!

Consider Figure 1 depicting a simplified soda can. If you leave just the right amount, you can get it to balance on the beveled edge, as seen in Figure 2. Try it next time you open a can (not right now).

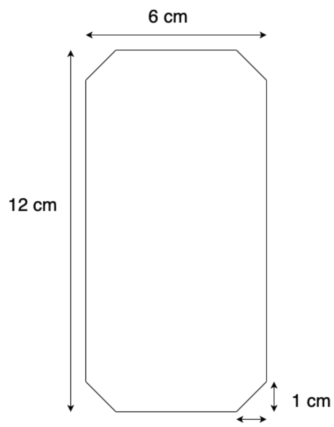


Figure 1

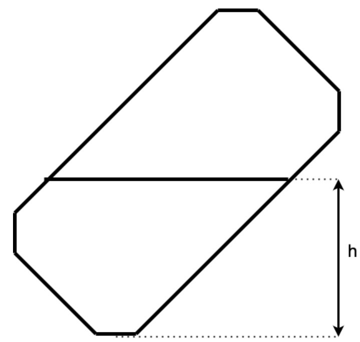


Figure 2

Problem 15:
See Figure 2. Let's take the can to be massless and initially empty. Let's also assume that we live in two dimensions. We start slowly filling it up with soda to a vertical height h. What is h just before the can tips over?

Solution

Similar to our solution to Problem 13, we draw a vertical line from the desired balancing point and split the regions into triangles and rectangles. Using symmetry and simple trigonometry, we find that:
$h = \sqrt{\sqrt{2} + \frac{8}{9}} + 3\sqrt{2}$
I'll try to write a more detailed solution later. Otherwise, just ask me.

Problem 16:
Think about how you might approach the previous problem with an actual three-dimensional soda can. Does h become larger or smaller?

Solution

This is a pretty open-ended question and is meant simply to make students think about how the problem would change in 3D. I believe that h gets smaller.

So far we've made the assumption our shapes have mass that is *uniformly distributed*. But that doesn't have to be the case.

Problem 17:
A mathemagical wizard will give you his staff if you can balance it horizontally on your finger. The strange magical staff has unit length and it's mass is distributed in a very special way. It's density decreases linearly from λ_0 at one end to 0 at the other. Where is the staff's balancing point?

Solution

This problem is really just Problem 13 again in disguise. So, the balancing point is at 1/3 the length of the staff measured from the dense-end.

Part 3: Pappus's Centroid Theorem

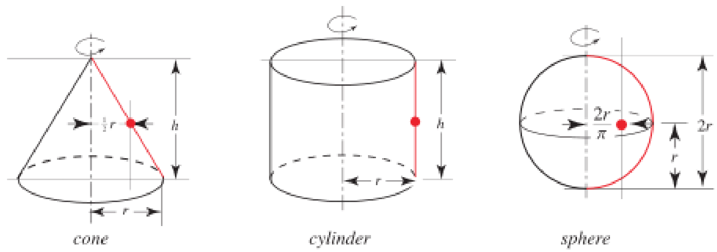


Figure 3

Figure 3 depicts three different surfaces constructed by revolving a line segment (in red) about a central axis. These are often called *surfaces of revolution.*

Pappus's First Centroid Theorem allows you to determine the area of a surface of revolution using information about the line segment and the axis of rotation.

Problem 18:
Can you intuitively come up with Pappus's First Centroid Theorem for yourself? Figure 3 is very helpful. Feel free to ask for any surface area formulae that might be helpful. What limitations are there on the theorem?

Solution

`https://en.wikipedia.org/wiki/Pappus%27s_centroid_theorem`
`https://mathworld.wolfram.com/PappussCentroidTheorem.html`

Pappus's Second Centroid Theorem simply extends the first theorem to *solids of revolution*, which are exactly what you think they are.

Problem 19:
Now that you've done the first theorem, what do you think Pappus's Second Centroid Theorem states?

Problem 20:
The centroid of a semi-circular line segment is already given in Figure 3, but what about the centroid of a filled semi-circle? (Hint: For a sphere of radius r, $V = \frac{4}{3}\pi r^3$)

Problem 21:
See Figure 4. Given arc AB with radius r and subtended by 2α, determine OG, the distance from the center of the circle to the centroid of the arc.

Solution

`https://mathspanda.com/A2FM/Lessons/Centres_of_mass_of_standard_shapes_LESSON.pdf`

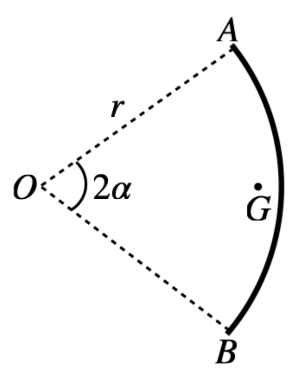


Figure 4

Problem 22:
Where is the centroid of the *sector* of the circle in Figure 4. (Hint: Cut it up.)

Problem 23:
Seeing your success with his linear staff, the wizard challenges you with another magical staff to balance. It looks identical to the first one, but you're told that the density decreases from λ_0 to 0 according to the function $\lambda(x) = \lambda_0\sqrt{1-x^2}$. Find the balancing point.

Solution

This is equivalent to finding the x-coordinate of the centroid of a quarter-circle. See Problem 22.

Problem 24:
Infinitely many masses m_i are placed at x_i along the positive x-axis, starting with $m_0 = 1$ placed at $x_0 = 1$. Each successive mass is placed twice as far from the origin compared to the previous one. But also, each successive mass has a quarter the weight of the previous one. Find the CoM if it exists.

Solution

We have $m_i = 1/4^i$ and $x_i = 2^i$ so

$$\begin{aligned}\sum_{i=0}^{\infty} m_i x_i &= \sum \frac{2^i}{4^i} \\ &= \sum \frac{1}{2^i} \\ &= \frac{1}{1-1/2} = 2 \qquad \text{as this is just a geometric series}\end{aligned}$$

$$M = \sum m_i = \frac{1}{1-1/4} = 4/3$$

Then

$$x_{CM} = \frac{\sum m_i x_i}{M} = 3/2$$

Problem 25:
(Bonus) Try to actually find h from Problem 13. Good luck.