LAMC Intermediate Group **October 18, 2015**

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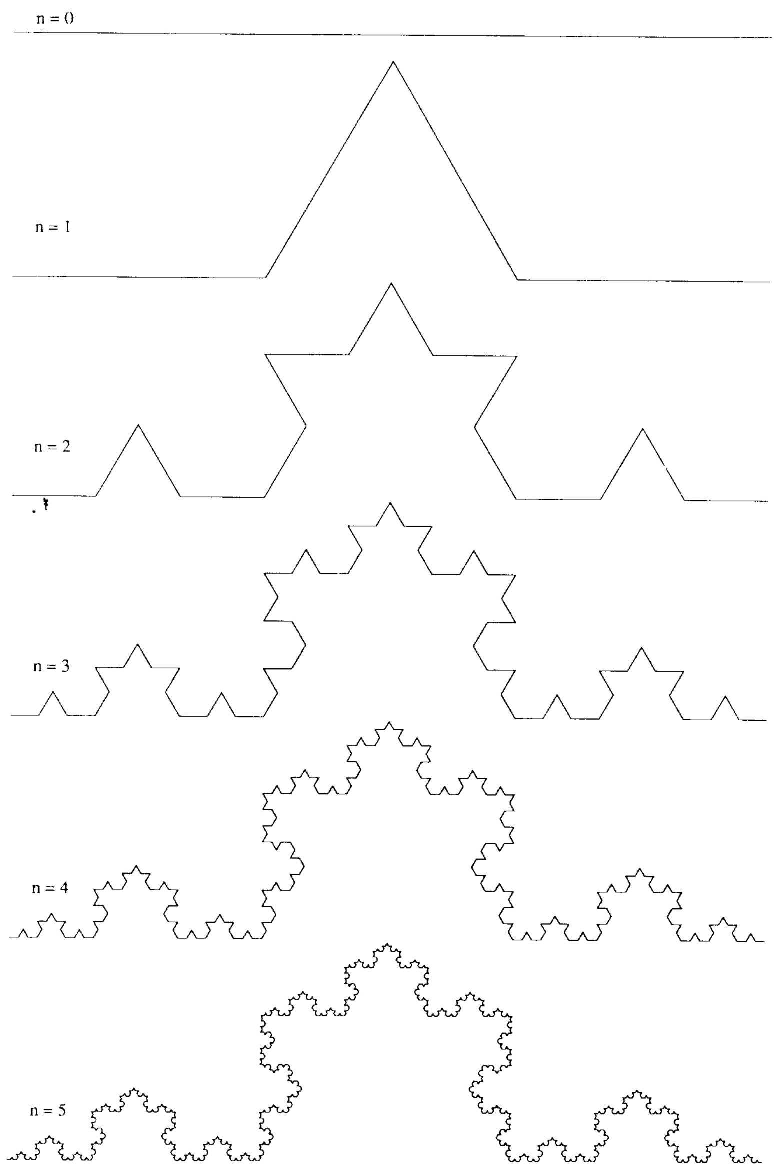
Recursive Functions and Fractals

In programming, a function is called *recursive*, if it uses itself as a subroutine.

Problem 1 *Give an example of a recursive function different from the one used in Problem 2 below.*

Problem 2 *Use Python to write a function factorial(n) that calls itself recursively to compute* $n!$ *for non-negative integers* n*.*

The limit of the following procedure is known as the *Koch curve.* The finite steps of the construction are known as the Koch curves of order $n = 0, 1, 2, \dots$



Assume that the length of the base segment used for the Koch curve construction is a.

Problem 3 *Let us denote l_n the length of the Koch curve or order n. Given $l_0 = a$, find the following.*

- $l_1 =$
- $l_2 =$
- $l_3 =$
- $l_4 =$
- $l_5 =$

The length l of the Koch curve is the limit of the sequence l_n as n tends to infinity.

$$l = \lim_{n \to \infty} l_n$$

What do you think l is?

If needed, use the Python Math module to answer the following questions.

Problem 4 *For what order n of the Koch curve of the base length $a = 10$ would the length l_n exceed*

- *1,000?*
- *1,000,000?*
- *700,984,371?*
- *55,000,000,000?*
- *A huge positive number M?*

Definition 1 *If for any positive number M, no matter how huge, there exists a number N, that depends on M, such that for any $n \geq N$ all the elements of the sequence l_n exceed M ($l_n > M$), then we say that the limit of the sequence l_n as n tends to infinity equals positive infinity.*

$$\lim_{n \to \infty} l_n = +\infty$$

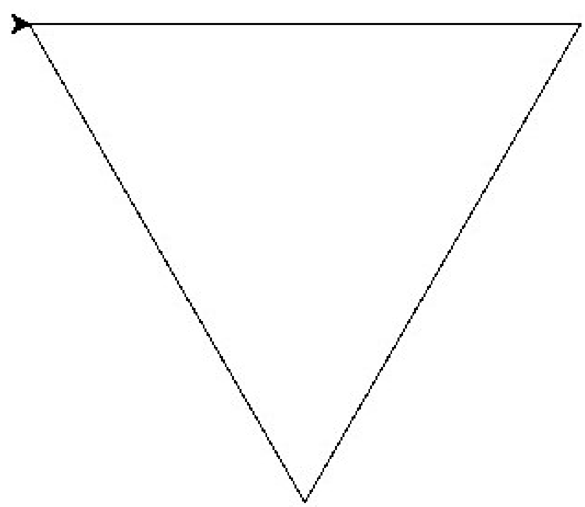
Problem 5 *Generalize your solution of Problem 4 above to any base length* $a > 0$ *to prove that any Koch curve has infinite length.*

Recall that to use the Turtle module commands without the *turtle.* beginning, we employ the following prompt.

>>> *from turtle import **

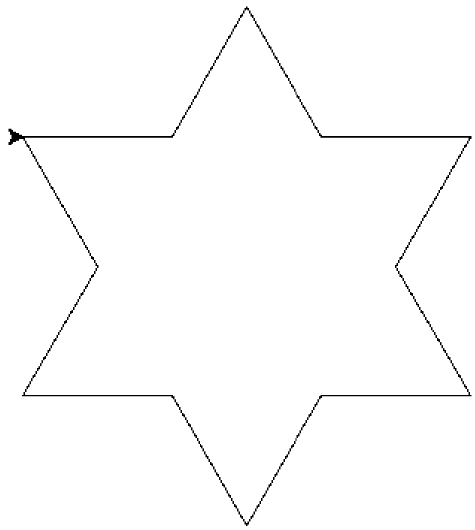
Problem 6 *In the Turtle module, write a function kochc(n,a) that draws the Koch curve of order* n *and base length* a*. Hint: recursion will help!*

The following equilateral triangle KS_0 of side length a (drawn using the Python's Turtle) is the base step for constructing a beautiful closed curve known as the *Koch snowflake*.



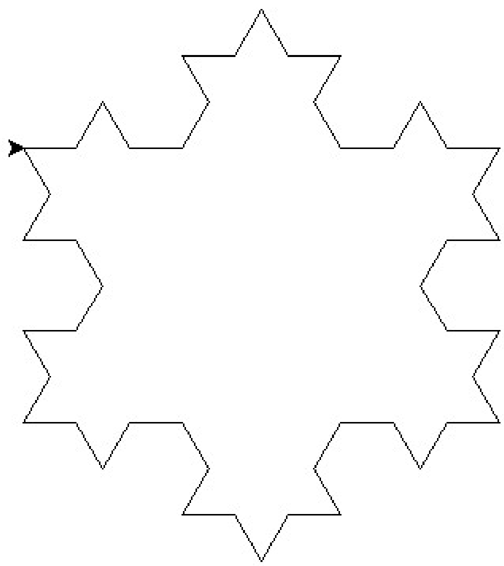
Problem 7 *Find the perimeter p_0 and area A_0 of KS_0.*

The figure KS_1 is the next step of the construction.



Problem 8 *Find the perimeter p_1 and area A_1 of KS_1.*

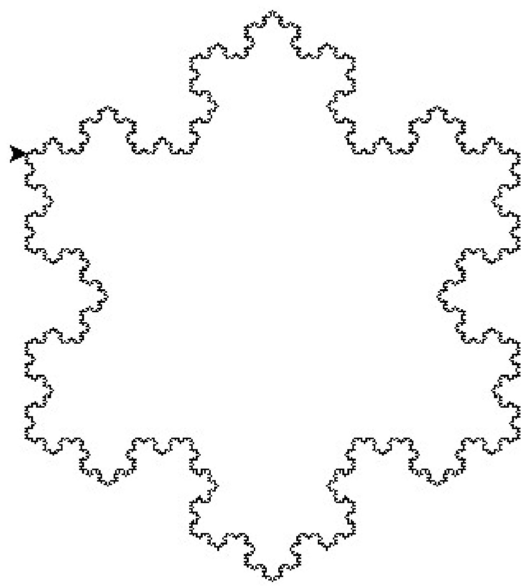
The figure KS_2 below is the next step.



Problem 9 *Find the perimeter p_2 and area A_2 of KS_2.*

Continuing the construction steps to infinity, we get the Koch snowflake.

$$KS = \lim_{n \to \infty} KS_n$$



Let $p = \lim_{n \to \infty} p_n$ and $A = \lim_{n \to \infty} A_n$ be the perimeter and the area of the Koch snowflake.

Problem 10 *Find p and A.*

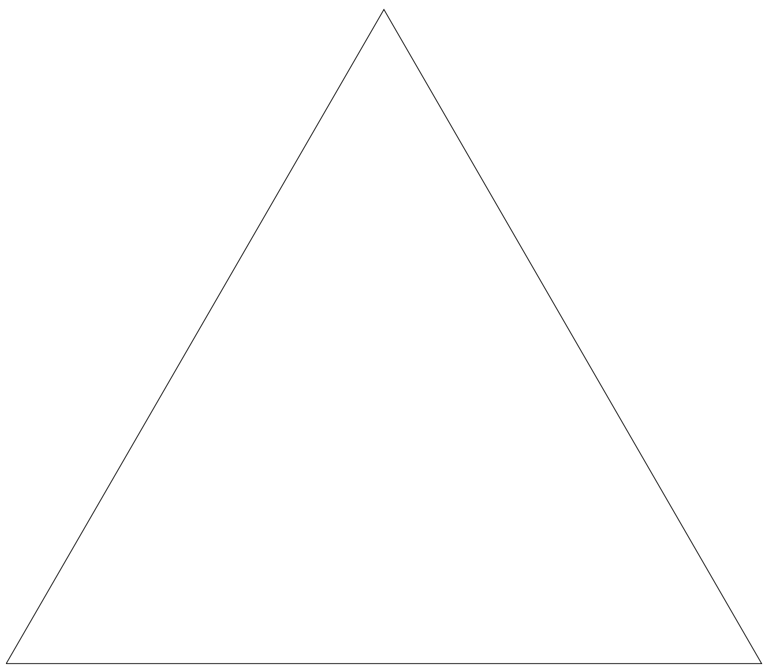
Problem 11 *In the Turtle module, write a function kochs(n,a) that draws the Koch snowflake of order n and base length a. Hint: use the function kochc(n,a)!*

Question 1 *What do you think is the dimension of the Koch curve and Koch snowflake?*

Below, we shall construct another famous fractal, called the *Sierpinski triangle* (a.k.a. the Sierpinski carpet, Sierpinski sieve, and Sierpinski gasket) after its inventor, Waclaw Sierpinski.[1]

Step 1

Let us take an equilateral triangle ST_0 of side length a.

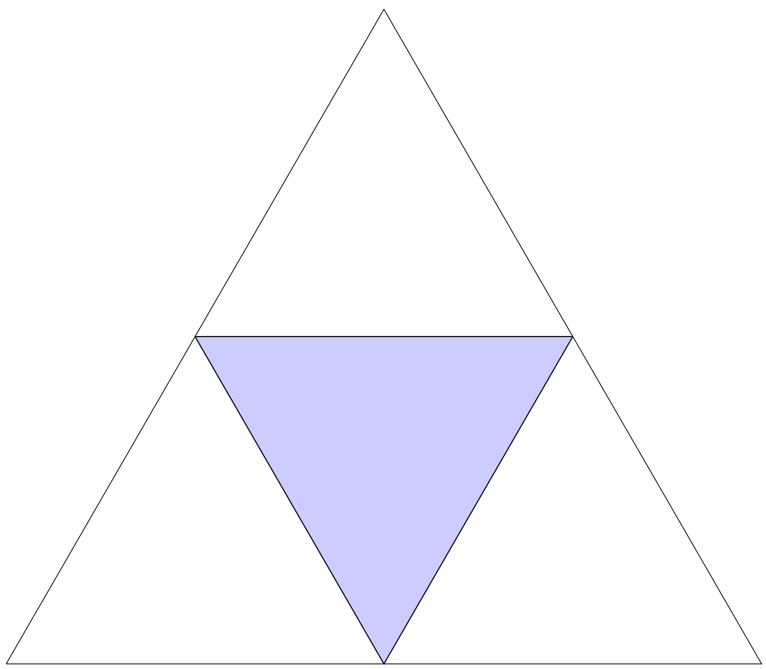


Problem 12 *Find the area* A_1 *of* ST_1.

[1] 1882 – 1969, a renown Polish mathematician.

Step 2

Let us take ST_1 and connect its midpoints. That splits the triangle into four smaller triangles of equal size. Let us cut out the one in the middle. Let us call ST_2 the resulting triangle with a triangular hole in the center.

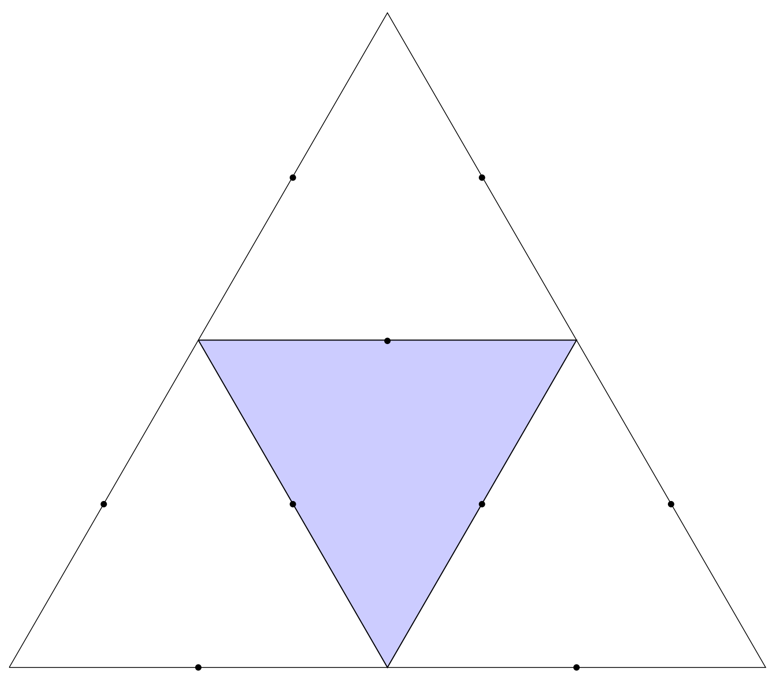


Problem 13 *Find the area A_2 of ST_2.*

Step 3

Let us take ST_2 and connect the midpoints of the three triangles it is made of. That splits each of the three triangles into four smaller triangles of equal size. Let us cut out the middle ones and call the resulting figure ST_3.

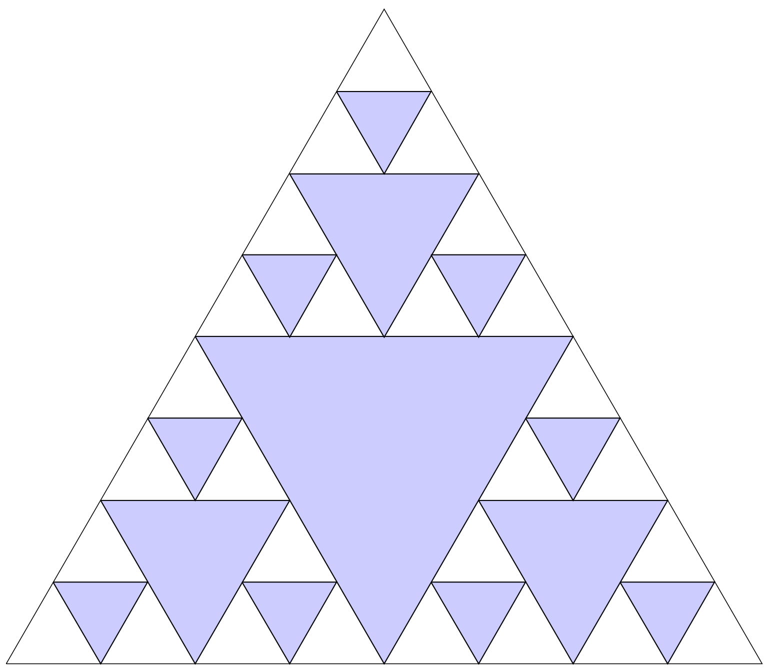
Problem 14 *Draw ST_3. Shade the triangles we need to cut out.*



Problem 15 *Find the area A_3 of ST_3.*

Step 4

Let us take ST_3 and connect the midpoints of the nine triangles it is made of. That splits each of the triangles into four smaller triangles of equal size. Let us cut out the middle ones and call the resulting figure ST_4.

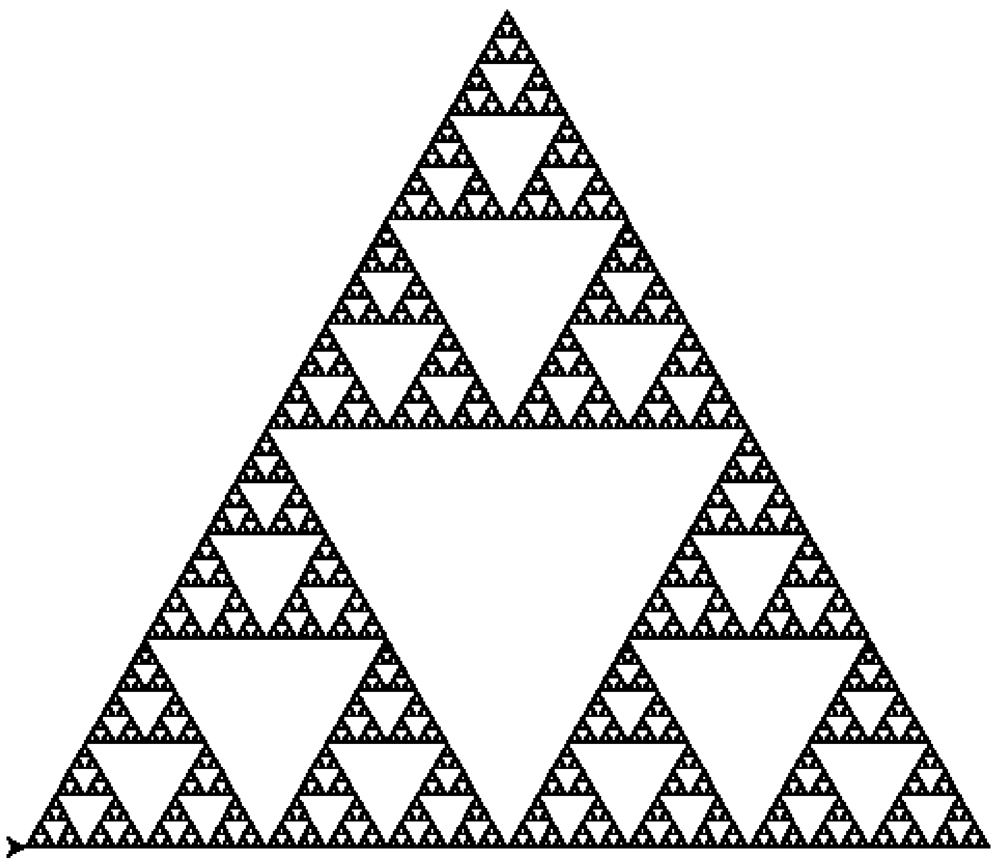


Problem 16 *Find the area A_4 of ST_4.*

Finishing the construction

Imagine that we take Step 5, Step 6, and so on to infinity. The resulting figure is called the Sierpinski triangle, or ST.

$$ST = \lim_{n \to \infty} ST_n$$



Problem 17 *For what order* n *of the Sierpinski triangle* ST_n *having the base length* $a = 10$ *would the area* A_n *become smaller than*

- *0.001?*

- *0.000,001?*

- *0.000,000,000,055?*

- *A tiny positive number* ϵ*?*

Definition 2 *If for any positive number* ϵ*, no matter how tiny, there exists a number* N*, that depends on* ϵ*, such that for any* $n \geq N$ *all the elements of the sequence of positive numbers* A_n *are less than* ϵ *(*$0 < A_n < \epsilon$*), then we say that the limit of the sequence* A_n *as* n *tends to infinity equals zero.*

$$\lim_{n \to \infty} A_n = 0$$

Problem 18 *Generalize your solution of Problem 17 above to any base length* $a > 0$ *to prove that the area of any Sierpinski triangle is zero.*

Question 2 *What do you think is the dimension of the Sierpinski triangle? How would you define the dimension of a geometric figure in general?*

Problem 19 *In the Turtle module, write a function st(n,a) that draws the Sierpinski triangle of order* n *and base length* a*.*

Problem 20 *Use recursion and the Turtle module to draw a beautiful fractal of your own.*

Next time, we will have a beauty contest among our fractals! To win, you will have not only to show the class a breathtaking picture, but also to present the Python code that generates it and to explain how it does the job.

If you are finished doing all the above, but there still remains some time...

The following problem was communicated to me by one of our students, Arul Kolla.

Problem 21

By putting suitable signs + , − , × , ÷ , (,) between the digits 3 3 3 3 many numbers can be generated, for example, (3 + 3) x 3 + 3 = 21.
Which of the following numbers can be generated this way?

a. ○17
b. ○31
c. ○54
d. ○22
e. ○60
f. ○73
g. ○90