Probability

Useful formulas

- $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$
- The number of ways to choose k objects from n is $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Basics

1. Non-transitive Dice
   Consider four dice A, B, C and D with faces given below:
   A: $4, 4, 4, 4, 0, 0$
   B: $3, 3, 3, 3, 3, 3$
   C: $6, 6, 2, 2, 2, 2$
   D: $5, 5, 5, 1, 1, 1$
   Suppose you play a game with your friend where you both pick a different die and then roll to see who gets the higher number. Should you be the first or second player to pick the die? What is your strategy? (Hint: Compute $P(A > B), P(B > C)$ and so on)

2. Flipping coins
   Two players alternatively toss pennies, and the first one to toss a heads wins. What is the probability that
   (a)The game never ends?
   (b)Player 1 wins?
   (c)Player 2 wins?
   (d)Devise a game using an ordinary fair coin for which the probability of winning is any real number x between 0 and 1. (Hint: Use the binary expansion of x)

3. Three people Alice, Bob and Charlie fight in a 3-way piston duel. All three know that Alice has a 0.3 probabliity of hitting her target, Bob never misses and Charlie has a 0.5 probability of hitting his target. Alice goes first, and the players proceed in cyclic order. If a player is shot, they don't shoot. What is Alice's strategy?

Expectation

4. What is the expected number of times you need to roll a fair die to get a 6? (Hint: First show that $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$)

5. What is the expected number of heads when you toss a fair coin 100 times?

6. What is the probability of getting exactly 50 heads in the above experiment? (You can use Stirling's approximation $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$ to approimate your answer by hand)

7. A deck of n cards which contains 3 aces is shuffled at random. The cards are turned over one at a time until the second ace appears. Show that the expected number of cards that are turned over is $\frac{n+1}{2}$. (Hint: Use symmetry)

Lattice paths and random walks

8. How many lattice paths are there from $(0, 0)$ to (a, b)?

9. How many lattice paths from $(0, 0$ to (n, n) do not go above the line $y = x$? (Hint: Show that the number that do cross that line is $\binom{2n}{n-1}$)

10. Suppose a drunk leaves a bar and walks up and down the street at random. We model the process by a random walk. We suppose he starts at 0 on the number line. Every second, he takes a step to the left or right with equal probability.
(a) What is the probability he is back at 0 after $2n$ steps?
(b) Given that he is back after $2n$ steps, what is the probability that this is his first time back?

11. Suppose that there is a cliff at 3 and his home is at -2. What is the probability that he reaches home safely?

Continuous probability

12. If a chord is selected an random on a fixed circle, what is the probability that its length exceeds the radius of the circle?

13. Alice and Bob agree to meet at the bus stop between 12 noon and 1pm. They each arrive at a random time in that interval and wait for 5 minutes before leaving, unless they see the other person. What is the probability that they meet each other?

14. (Buffon's needle) An infinite table has inscribed on it parallel lines spaced 2 units apart. A needle of length 2 is twirled and tossed on the table. What is the probability that it crosses a line? (This requires calculus; Answer: $\frac{2}{\pi}$)