Geometry and perspectives

Terry Wang

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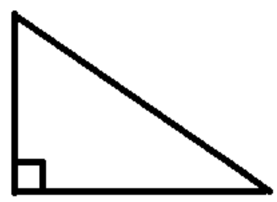
For the duration of this packet, all * problems are the more difficult ones.

1 Introduction to geometry

1.1 Pythagoras and right triangles

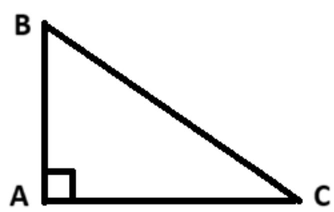
Many of you may already be familiar with the concept of a right triangle, specifically, a triangle is known as a **right triangle** if one of its angles is a **right angle** or 90 degrees:



As a side note, from here on outwards the Δ symbol will denote a triangle, $|XY|$ will denote the distance between two points X and Y, and $\overline{XY}$ is the line itself spanning X and Y. And for two lines L and K, I will denote their intersection $L \cdot K$ (if it exists). The most important theorem regarding these triangles is the **Pythagorean theorem** (not discovered by Pythagoras, funnily enough):

Theorem 1.1 Let ΔABC be a right triangle with the angle at A the 90 degrees angle. Then:

$$|AB|^2 + |AC|^2 = |BC|^2$$

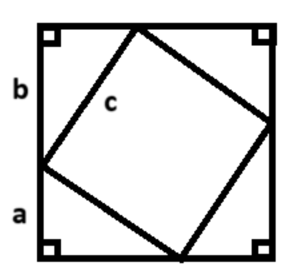


Or, written in the easier to remember form, $a^2 + b^2 = c^2$, where a, b are the short sides of the right triangle and c is the longest side, called the **legs** and the **hypotenuse** respectively.

Problem 1.1 Suppose that a right triangle has side lengths $3, 4$, suppose you also know that the third side length is the longest one. What is the length of this third side?

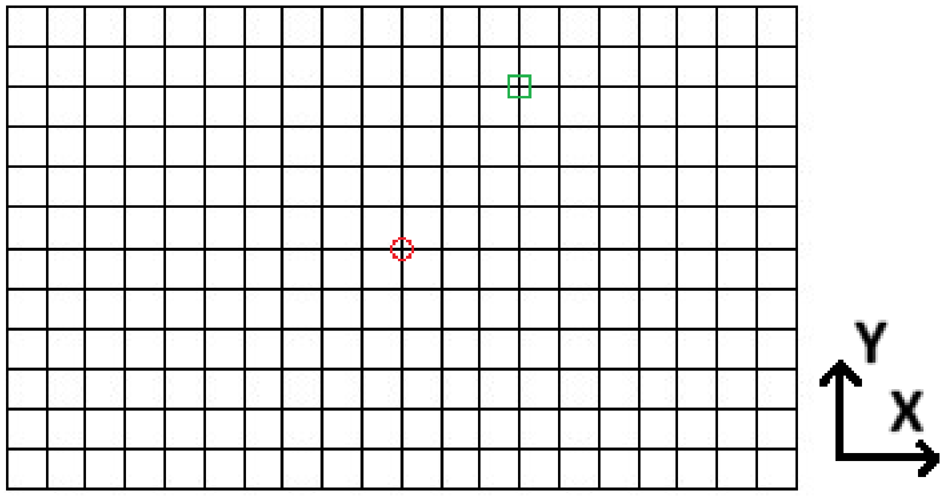
Problem 1.2 Suppose that a right triangle has side lengths $12, 13$, suppose you also know that the third side length is the shortest one. What is the length of this third side?

Problem 1.3 Prove the Pythagorean theorem by considering the following diagram:



1.2 Coordinates

For the future, it will often be helpful to describe points using coordinates. Suppose we have the following grid also known as the **Cartesian plane**:



Let's call the circled point the **origin**. And suppose we want to supply each grid point (intersection of grid lines) with a pair of numbers called its **coordinates**, then, one way to do so and the way we will do so is by specifying its relation to this origin point.

First of all, as the arrows in the above diagram suggest, right for the time being will be the x direction and up will be the y direction.

For instance we will denote $(3, 4)$ as the point that is 3 units in the x direction and 4 units in the y direction from the origin (boxed point in the above diagram). And in general, (a, b) means a units in the x direction and b units in the y direction from the origin.

Problem 1.4 Graph the following points: $(1, 2), (0, 1), (4, 0), (0, 0), (-1, 3), (2, -4)$ in the above diagram or until you feel like you know what's going on.

In our notation, the first number in the coordinate, will be the x **coordinate**, and the second will be the y **coordinate**. The points with 0 in the x coordinate will be called the y **axis** and the points with 0 in the y coordinate will be called the x **axis** (backwards, yes I know).

Problem 1.5 Draw the x and y axis in the diagram above.

Suppose now we wanted to find the distance between two points on the coordinate plane, say (a, b) and (x, y).

Problem 1.6 Derive the following distance formula: the distance between to points on the $x - y$ plane, say (a, b) and (x, y) is:

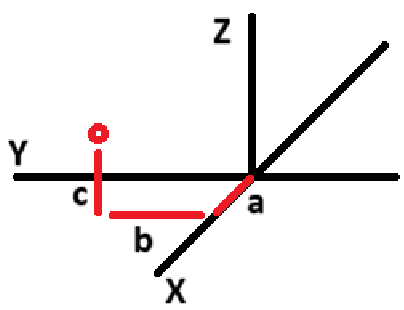
$$\sqrt{(x-a)^2 + (y-b)^2} = d$$

HINT. Pythagorean theorem.

1.3 3D coordinates

Now we obviously live in a 3D world rather than a 2D one, and so it is often desirable to have a system to describe the location of points in 3D with numbers as well. We will adopt the following convention:

1. First we need a suitable choice of an origin, which will have the coordinate $(0, 0, 0)$.
2. We need to choose x, y, z directions, these directions must be such that if one drew lines in the x, y and z directions passing through the origin, they would all meet at right angles.
3. Now given a triplet of points (a, b, c), this will be the point that is a units in the x direction, b units in the y direction, and c units in the z direction from the origin.



In the above diagram, the intersection of the three black lines is the origin. The first and second numbers in the triplets are still called x and y coordinates respectively, but the third coordinate will be the z **coordinate**. The lines labeled X, Y and Z are also, respectively, the x, y, **and** z **axis**. These can also be described as the points where $y = z = 0$ (shorthand for the set of points where the y, z coordinates are 0), $x = z = 0$, and $x = y = 0$ respectively.

The $x - y$ **plane** is the plane such that $z = 0$, the $y - z$ **plane** is the plane such that $x = 0$, and the $z - x$ **plane** is the plane such that $y = 0$.

Problem 1.7 What angles do the $x - y$, $x - z$, and $y - z$ planes meet at?

Problem 1.8* Let (a, b, c) and (p, q, r) be the coordinates of two points in 3D. Prove that the distance between them is:

$$d = \sqrt{(a-p)^2 + (b-q)^2 + (c-r)^2}$$

2 Projections and perspective

Projective geometry is a portion of geometry that goes untaught for the most part in highschool and university math education. Yet the idea of projective spaces does eventually come up in various topics of higher mathematics (albeit developed in a way that is much different from the treatment we will have here) and even in some popular math.

Projective geometry originated from considerations in painting: how does one draw a 3D scene onto a 2D canvas? There are a couple ways of doing so and any art student might tell you some of them, but here we are going to deal with the simplest method.

2.1 Ray-tracing and the equation of a 3D line

As will become immediately obvious why, we will also need a way to describe lines in 3D as well. For this purpose, we will say that a curve in 3D has equation $l(t) = (l_x(t), l_y(t), l_z(t))$ if for every real number value of t, the coordinate $(l_x(t), l_y(t), l_z(t))$ lies on the desired curve and for every point on the curve, there exists a value of t such that $(l_x(t), l_y(t), l_z(t))$ is this point.

Problem 2.1 What is the equation of the line passing through the points $(1, 0, 0)$, $(0, 0, 1)$, find the intersection of this line with the line $k(t) = (1, t, t)$.

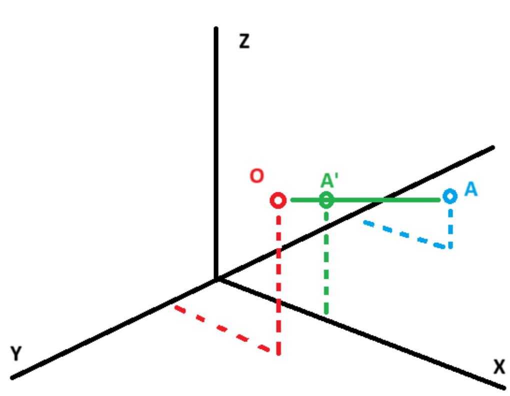
Problem 2.2 What is the equation of the line that is the x axis.

Problem 2.3* Given two distinct points $P = (a, b, c)$ and $O = (p, q, r)$ prove that:

$$l(t) = (at + (1-t)p, bt + (1-t)q, ct + (1-t)r)$$

Is an equation of a line passing through these two points.

Consider the following diagram:



Suppose you were standing at a point O. Let A be some point and let A' be the the intersection of the line $\overline{OA}$ and the $x - z$ plane just as above. From where you are standing, the points A and A' would appear the same, as in they would overlap in your vision. As such, imagine the $x - z$ plane as a computer screen, if a programmer wanted to simulate the experience of seeing a point at A, then they would command the computer to render the point A' on the screen.

The point O is known as the **viewing point**, the line connecting A, A' and O is known as the **projecting line** and A' is known as the **image** of A. The $x - z$ plane, in this case, is known as the **picture plane** and the $x - y$ plane is the **object plane**. This process (starting with a point and calculating the image on the $x - z$ plane) is known as the **perspective transformation**.

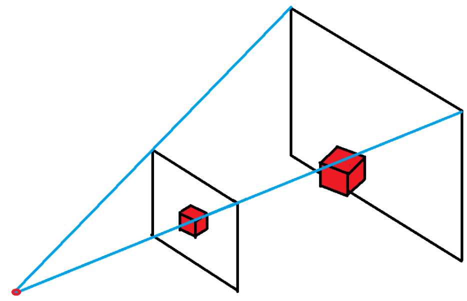
As a side note, we will not consider the case where the viewpoint is on the object plane, so the point O above must have a nonzero z coordinate.

Problem 2.4 Suppose you are standing at the point $(0, -3, 2)$, the with the above setup (same object and picture plane). What is the image of the point $(2, 1, 0)$?

Problem 2.5 Now try 2.4 again except with the picture plane now being the plane $y = 1$.

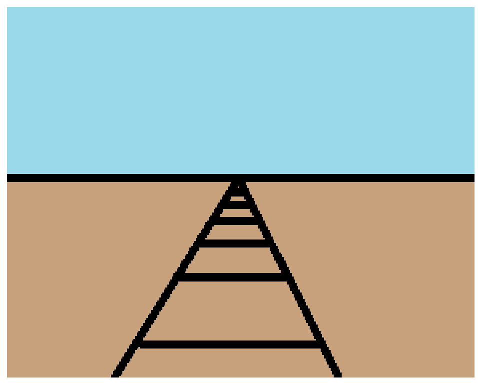
Problem 2.6* Now try 2.4 again except with the circle (in the $x - y$ plane) with the equation $x^2 + (y - 2)^2 = 1$. As in, find the equation of the resulting image of the circle from the perspective transformation described in 2.4. What type of shape is this image?

In general, a lot more mathematics goes into how computers display the images we see (shading, which shapes are covered and not displayed, etc.), but since we are aiming to use the idea of projection to solve problems, we won't get into it. In fact, from here on out, we will restrict ourselves to only work with projecting objects which lie in the $x - y$ plane. Also for our purposes, we will assume the picture and object planes to be respectively the $x - z$ and $x - y$ planes respectively for standardization.

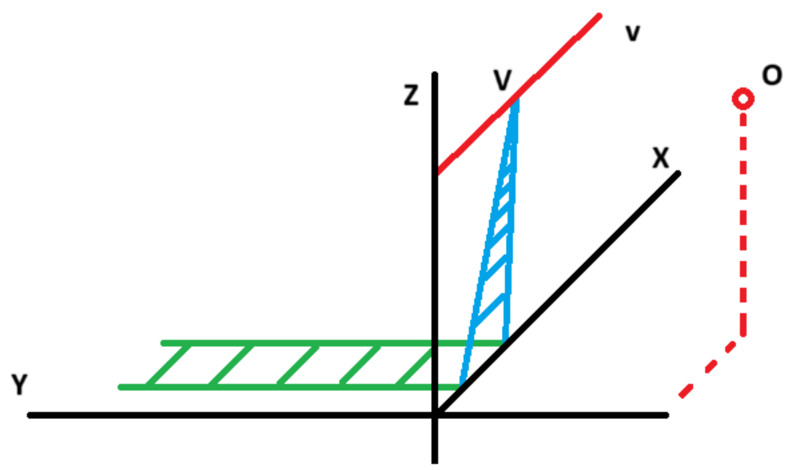


2.2 Lines meeting at infinity

Suppose you were standing in a large flat desert on train tracks that stretched into the horizon. The train tracks, to you, would seem to intersect on the horizon:



Suppose now you were instead rendering this scene on a computer screen, then this point is called the **vanishing point**:



In the above diagram, the train tracks themselves are in the object plane (green), and the image (blue) is in the picture plane, intersecting at V, the vanishing point. The line, parallel to the x axis and going through V is known as the **vanishing line** or the horizon (labeled v). One thing I have deliberately glossed over is the fact that the image of a line is a line itself. This is not entirely trivial to prove, but also not in the spirit of our discussion, so we are skipping the proof.

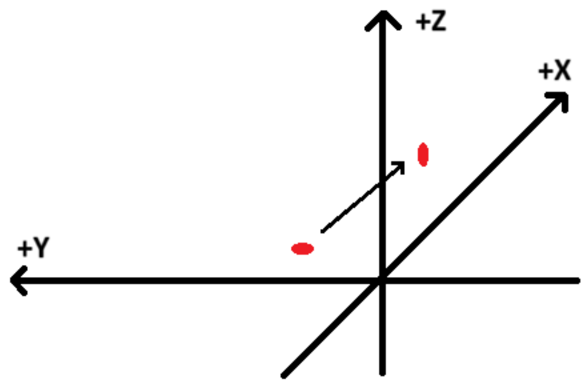
Theorem. The image of a straight line in the object plane is a straight line in the picture plane under a perspective transformation.

Problem 2.7 I've implicitly made a lot of claims about the vanishing point in the above diagram. Prove the following claims:

1. All lines parallel to the y axis in the object plane appear to vanish at the same point V.
2. The vanishing point can be constructed by taking the intersection of the $x - z$ plane and the line, perpendicular to the $x - z$ plane, and passing through O.
3. Do other parallel lines (not necessarily parallel to the y axis, just to each other) also appear to intersect on the vanishing line (to a viewer at point O)?

2.3 Rotating the plane

We will now work on abstracting our previous ideas. First of all, we will now consider an additional transformation. Let R be the transformation that rotates the object plane 90 degrees along the x axis (clockwise when viewed from the $-x$ axis) so that it coincides with the picture plane:



Recall that in our standard setup, the $x-y$ plane is the object plane and the $x-z$ plane is the picture plane.

Let A be the original point in the object plane, and let us call the new point $R(A) = A^*$. A natural question to ask is when is this A^* equal to its perspective transformed counterpart $A \to A'$ (with respect to some viewpoint)? Note that if A is on the x axis then we have equality, since neither transformation does anything. As such we also call this axis the **invariant line** or **axis** of the perspective transformation.

Problem 2.8 With the standard setup (see the above figure) of object and picture plane, what is the image, A^*, of the point $(1, 1, 0)$.

Problem 2.9 What is the general coordinates of the image point point $(x, y, 0)$ under the rotation transformation?

Problem 2.10 Show that there is at most one point where $A^* = A'$ that is not on the x axis, this point is known as the **center**.

Problem 2.11* Show that under the $A^* \to A \to A'$ transformation, ie. apply R^{-1} and then calculate the perspective transformation, any line passing through the center O^* gets mapped to itself.

Suppose now we are given A^* and we are asked to calculate A' (with respect to some viewpoint that we do not know). Suppose we also know the locations of two points: B^* gets transformed into B'; and also we are given the location of the center O^* and the invariant line l (on the $x - z$ plane). Without actually knowing the viewpoint, we can in fact calculate A' as follows:

1. We know that the line $\overline{A^*O^*}$ is invariant and so A' lies on $\overline{A^*O^*}$.
2. Extend $\overline{A^*B^*}$ to intersect l at a point L. Now we know that $\overline{A^*B^*} = \overline{LB^*} \to \overline{LB'}$. Now $\overline{A^*B^*} \to \overline{A'B'}$ as well so we must have that A' is on $\overline{LB'}$.
3. Thus A' is the intersection of $\overline{A^*O^*}$ and $\overline{LB'}$.

Of course it may happen that the above two are parallel, in which case there is no image point A'.

Problem 2.12* Call this line with no defined image S, what is the image of S under the reverse rotation transformation?

3 Abstract projections

With this we now define the abstract definition of a perspective projection, the idea is to mimic the process of $A^* \to A \to A'$ as described previously but without mentioning 3 dimensions, instead only working in the plane.

A **plane perspective** transformation with **axis** l, **center** B, and **base points** $B \to B'$ (ie. we know that B gets mapped to B') is a transformation such that a point A is mapped by the following series of steps:

1. Connect A and B and extend this line to intersect l at a point L.
2. Connect L and B' and extend to intersect the line $\overline{OA}$ at the point A'.
3. If the previous step fails then there is no image of A.
4. If the first step fails, first pick some point C for which steps $1-2$ works and such that $\overline{AC}$ is not parallel to l, compute C' and use C, C' in place of B, B'.

Problem 3.1 Suppose we have two plane perspective transformations, denoted S, T with the same axis and center but such that $B \to B'$ are the base points of S but $B' \to B$ are the base points of T. Show that S and T are such that if S is defined on a point A, then $T(S(A)) = A$.

Problem 3.2* Let G, G', H, H', J, J' be six points such that $\overline{GG'}, \overline{HH'}$, and $\overline{JJ'}$ are all concurrent at a point O. Show that there exists a plane perspective transformation T such that $T(G) = G', T(H) = H'$, and $T(J) = J'$. You can assume that GH and $G'H'$ are not parallel.

Problem 3.3 (Desargues) Prove the following statement. Let ΔABC and ΔXYZ be two triangles, suppose that $\overline{AX}, \overline{BY}, \overline{CZ}$ are concurrent, prove that the points defined by the intersections $\overline{AB} \cdot \overline{XY}, \overline{BC} \cdot \overline{YZ}$, and $\overline{AC} \cdot \overline{XZ}$ are colinear.

HINT. Use problem 3.2.

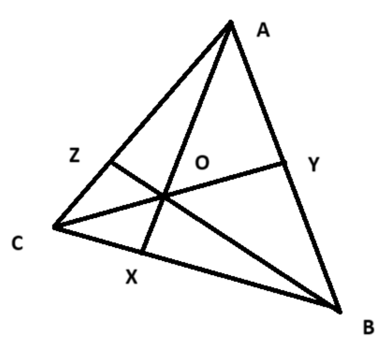
Problem 3.4 Let ABC be three colinear points and let T be a plane perspective transformation with axis l such that $T(A), T(B), T(C)$ are all defined. Suppose that $\overline{AC}$ is parallel to l, show that $T(A)T(B)T(C)$ are colinear and form a line parallel to $\overline{AC}$, and furthermore:

$$\frac{|AB|}{|BC|} = \frac{|T(A)T(B)|}{|T(B)T(C)|}$$

Problem 3.5 Let L and l be two parallel line segments, with only a straight edge (no compass, and no markings on the straight edge), figure out a way to partition one of the line segments into two equal halves. Here is a diagram for you to draw on:

Now, justify (prove) that what you just came up with is indeed a bisection of one of the line segments.

Hint: You can use the following theorem that appears somewhat frequently in contest mathematics:
Theorem (Ceva). Let ΔABC be a triangle and O be a proper interior point (not on one of the edges), let X, Y, and Z be the points generated by extending $\overline{AO}$ to intersect with $\overline{BC}$ (respectively $\overline{BO}$ to $\overline{AC}$ and $\overline{CO}$ to $\overline{AB}$), then we have:



$$\frac{|AZ|}{|ZB|}\frac{|BX|}{|XC|}\frac{|CY|}{|YA|} = 1$$

Problem 3.6* This problem has a bit of unfortunate history to it. The Mathematics Department of Moscow State University used to be discriminatory against Jewish students, so they would purposely give extremely hard problems to them in oral entrance exams to purposely fail them. All of these problems were very simple to state, yet exceedingly hard to solve. This was one such problem.

Given two parallel line segments and a straight edge, devise a scheme to split one of the line segments into 6 equal line segments.

HINT. Use problems 3.4 and 3.5.