Week 1: Intro to the Collatz Conjecture and Iterated Functions

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Welcome (back) to the Math Circle! This week, we will explore the simple operation of applying a single function repeatedly. This is often called a *discrete dynamical system*. This basic idea has applications to number theory, differential equations, and applied math. We will explore one such application to number theory called the Collatz conjecture. This is one of the most famous unsolved problems in math. At the end, we will also use this technique to model the growth of a population.

In this packet, we will be working with functions f of positive natural numbers, integers, positive rationals, real numbers. The domain of the functions in this packet will be the set of all real numbers unless stated otherwise.

For any number x, we will denote by $f^2(x)$ the iterated function $f(f(x))$. Similarly, for any positive integer n, we define $f^n(x)$ as applying the function f n-many times to the number x.

Problem 1. Let $f(x)$ be the function $f(x) = 2x$. Compute the following

- $f^3(1)$
- $f^2(2)$
- $f^2(3)$

We say that the function f has a *fixed point* at the number x if $f(x) = x$.

Problem 2. Find the fixed points (if there are any) of the following functions.

- $f(x) = 2x$
- $f(x) = x^2 - x$
- $f(x) = 2/x$. The domain of this function is the set of all *nonzero* real numbers.

We say that the function f has a *cycle* at the number x if there exists a positive natural number n for which $f^n(x) = x$. The smallest n for which $f^n(x) = x$ is called the *length* of the cycle at x.

Problem 3. Explain why a cycle of length 1 is exactly the same thing as a fixed point of a function.

Problem 4. Find a cycle of length 2 for the function $f(x) = -x$. Also find a fixed point.

Problem 5. Now, find *all* of the cycles for $f(x) = -x$.

Problem 6. Find all of the cycles and their lengths for the following function.

$$f(x) = \begin{cases} x - 3 & \text{if } x \text{ is positive} \\ -x & \text{if } x \text{ is negative or zero} \end{cases}$$

For these kinds of problems, it is helpful to draw the *orbit diagram* of f. We draw an arrow from n to m if $f(n) = m$. We will similarly say that m is in the *orbit*, or *f-orbit* of n if $f^c(n) = m$ for some positive natural number c.

Problem 7. Draw the orbit diagram for the following function f, which is defined on all integers from 0 to 9.

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is not a multiple of 3} \\ 9-n & \text{if } n = 0, 3, 6 \\ n & \text{if } n = 9 \end{cases}$$

Use this diagram to identify all of the cycles.

We will now start to look at the Collatz conjecture. For the next few problems, we use the following function f defined on positive natural numbers.

$$f(x) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

Problem 8. Compute $f(n)$ for all n from 1 to 10.

Problem 9. Compute the following:

- $f^3(1)$
- $f^3(2)$
- $f^3(4)$
- $f^3(5)$
- $f^5(3)$
- $f^5(21)$

Draw the piece of the orbit diagram for f that includes these numbers. What do you notice?

You probably noticed in the last problem that for all of the natural numbers n that you tried, there is some natural number c_n for which $f^{c_n}(n) = 1$, or equivalently, 1 is inside the f-orbit of every positive natural number that you checked. The Collatz conjecture states that such a (finite) c_n exists for *all* positive natural numbers n. In other words, people think that no matter what positive number you start with, if you keep applying f to it, you'll *always* end up at 1. However, no one knows for sure if this is true! This conjecture has been verified for all n less than $\sim 3 \times 10^{20}$. What makes this problem so interesting to mathematicians is that despite how easy it is to state, no one has figured out a general proof. This means that we still have a lot to understand about even basic things.

Problem 10. For each n from 1 to 10, find the smallest c_n for which $f^{c_n}(n) = 1$.

From Quanta Magazine

Problem 11. Show that for the Collatz function f, 1 is in the f-orbit of infinitely many positive natural numbers.

Problem 12. Now, show that for the Collatz function f, 1 is in the f-orbit of infinitely many *odd* natural numbers.

There is one easy way to modify the Collatz function. For this problem only, let us now consider the function given by the same formula as f, but now defined on all integers. So, we now have to compute values like $f(0)$ and $f(-1)$.

Problem 13. Besides the cycle $4 \to 2 \to 1 \to 4$, find 3 cycles of f.

The behavior we saw in the Collatz function is pretty special. Specifically, the fact that we only know of one cycle is very interesting.

This problem due to Terry Tao, Quanta Magazine.

Problem 14. Consider the modified Collatz function

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n-1 & \text{if } n \text{ is odd} \end{cases}$$

There are three different cycles for this function. Find them all!

Bonus Problems

All of the problems from this page forward are bonus problems. Some of these may be much more difficult than were the previous problems, some are much more open-ended and conceptual, and some are just on unrelated material that you may find interesting. Feel free to skip around this section.

There is another way to extend the domain of f. For this problem only, consider the function given by the same formula as f, but now defined on all positive rational numbers in the following way. Any rational number can be written as a/b where a and b are positive integers, and a and b do not have any common factors. Then if a is even, we define $f(a/b) = a/2b$, and if a is odd, we define $f(a/b) = (3a+1)/b$.

Problem 15. The cycles of this function are closely related to the cycles of the ordinary Collatz function. What is the relationship?

Problem 16. Show that if the Collatz function f has any cycles besides $4 \to 2 \to 1 \to 4$, then the other cycles must be length at least 6. In fact, they must be length at least $186,265,759,595$, but you don't need to prove this.

The next few problems involve a very different function f, and different techniques will be used here.

For any real number r, we can define the function $f_r(x) := rx(1-x)$. These maps are often called "logistic maps". Functions like these can be used to model the growth of a population that has limited access to food. For realistic models, we will have $x \in [0, 1]$ and $r \in [0, 4]$. $x = 0$ means that there is no population, and $x = 1$ means that the population is maximum size. If x is too large, there will not be enough food for everyone, so there will be a smaller population next year. Conversely, if x is very small, then there will not be many babies next year either.

Problem 17. For each value of r, find the fixed points of f_r. Your answer should involve formulas in terms of r.

Problem 18. Suppose that $r \in [0, 4]$ and $x \in [0, 1]$. Show that $f_r(x) \in [0, 1]$

So, as long as $r \in [0, 4]$, we can view f_r as a function from $[0, 1]$ to itself. We will do so for the remaining problems.

Problem 19. Show that if $r \in [0, 1]$, then for any $x \in [0, 1]$, we have that $x \geq f_r(x) \geq f_r^2(x) \geq f_r^3(x)$.... Since this sequence is bounded below, one can prove that the graph of $f_r^n(x)$ vs n has a horizontal asymptote somewhere for any x. What do you think it is?

Problem 20. For $x = 1/4, 1/3$, and $1/2$, compute the first few values in the f_2-orbit of x. What do you notice? Can you prove that the graph of f_2 has a horizontal asymptote for all $x \in [0, 1]$?

Problem 21. For r large enough, f_r has cycles of length 2, i.e., pairs (x, y) with $x \neq y$ but $f_r(x) = y$ and $f_r(y) = x$. Which $r > 0$ have this property? What are the length-2 cycles in this case?