NIM

MATH CIRCLE (ADVANCED) 12/02/2012

Recall from last time the idea of dividing a game into **P-positions** and **N-positions.**

A P-position is one that is winning for the previous player (that is, the one who just moved) while a N-position is one that is winning for the next player.

Therefore, if a game starts in a P-position, Player II wins, while if the game starts in a N-position, Player I wins.

The game of Nim: Suppose we have k piles of stones, with n_1 stones in the first pile, n_2 stones in the second pile, etc. Player I and Player II take turns removing any number (at least one!) stone from a single pile. The player who takes the last stone wins.

We will write the above game as $(n_1, n_2, \ldots, n_k)-$Nim.

0) a) List the P-positions and N-positions for Nim played with only one pile ($k = 1$).

The only P-position is (0).

b) For what values of n_1 does Player I win?

Player I wins all games.

1) a) Play Nim with piles of size 6 and 4 (i.e. $(6, 4)-$Nim). Who wins?

Player I wins.

b) Explain the winning strategy.

Player I first moves to position $(4, 4)$. Then he plays so that both piles always have the same number after his turn.

2) a) Characterize the P-positions and N-positions for $(n_1, n_2)-$Nim.

(n_1, n_2) is a P-position if and only if $n_1 = n_2$.

b) For what pairs (n_1, n_2) does Player I win?

Player I wins $(n_1, n_2)-$Nim if and only if $n_1 \neq n_2$.

The Nim-sum (denoted by $\oplus$) of two numbers is computed as follows. Convert the numbers to binary, then add (in binary) WITHOUT carrying any digits, and finally convert back to decimal form. For example,

$$60 \oplus 47 \to \frac{\begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{array}}{\begin{array}{cccccc} 0 & 1 & 0 & 0 & 1 & 1 \end{array}} \to 10011_2 = 19.$$

3) Compute the following Nim-sums:

a) $6 \oplus 4$

2

b) $12 \oplus 12$

0

c) $25 \oplus 8$

17

d) $3 \oplus 4 \oplus 5$

2

e) $19 \oplus 60 \oplus 47$

0

f) $2 \oplus 2 \oplus 2 \oplus 2$

0

g) $5 \oplus 2 \oplus 2 \oplus 2$

7

4) Write out P-positions and N-positions for $(n_1, n_2, n_3)-$Nim for $n_1+n_2+n_3 \leq 6$. Think of ways to minimize the number of positions you need to write out!

First note that without loss of generality $n_1 \geq n_2 \geq n_3$ (why?). Further, if $n_3 = 0$ see 2)a). Therefore we also assume $n_3 > 0$.

The only P-position satisfying the above are: $(3, 2, 1)$.

The N-positions are $(1, 1, 1), (2, 1, 1), (2, 2, 1), (3, 1, 1)$.

5) a) Prove that $a \oplus b = 0$ if and only if $a = b$.

If $a = b$ all the binary digits are the same, so $a \oplus b = 0$. If $a \neq b$ then (at least) one pair of binary digits do not match, hence $a \oplus b \neq 0$.

b) Restate 2) in terms of Nim-sums.

(n_1, n_2) is a P-position if and only if $n_1 \oplus n_2 = 0$.

6) a) Characterize the P-positions and N-positions for $(n_1, n_2, n_3)-$Nim.

See 8).

b) For what triples (n_1, n_2, n_3) does Player I win?

See 8).

7)* Prove 6)a).

See 8).

8)* Characterize the P-positions and N-positions for $(n_1, n_2, \ldots, n_k)-$Nim. Prove your answer!

$(n_1, n_2, \ldots, n_k)$ is a P-position if and only if $n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$.

The result follows from the two statements:

i) If $(n_1, n_2, \ldots, n_k)$ is a P-position then any move results in a N-position.

ii) If $(n_1, n_2, \ldots, n_k)$ is a N-position then it is possible to move to a P-position.

i) is straightforward to prove using the Nim-sums.

For ii), suppose $n_1 \oplus n_2 \oplus \cdots \oplus n_k = x \neq 0$. Let i be the leftmost nonzero digit of x in its binary representation. Thus, for some j, the ith digit in the binary representation of n_j is also nonzero. Check that removing $n_j - (x \oplus n_j)$ from the jth pile works.

∞) Solve the following games (i.e. try to characterize the P-positions and N-positions and try to prove your answer. However, this may not be possible!).

a) l-Subtraction Game: Similar to $(n_1, n_2, \ldots, n_k)-$Nim, except players can remove at most l stones.

$(n_1, n_2, \ldots, n_k)$ is a P-position if and only if $n_1(\text{mod } l+1) \oplus n_2(\text{mod } l+1) \oplus \cdots \oplus n_k(\text{mod } l+1) = 0$.

b) Greedy Nim: Similar to $(n_1, n_2, \ldots, n_k)-$Nim, except players can only remove stones from the largest pile.

A position is a P-position if and only if there are an even number of largest piles.

c) Multiple Pile Nim: Similar to $(n_1, n_2, \ldots, n_k)-$Nim, except players can also remove the same number of stones from both piles.

Some small p-Positions when $k = 2$ and $n_1 \geq n_2$ are $(0,0), (2,1), (5,3), (7,4), (10,6)$.

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg “Mathematical Circles (Russian Experience)”