Problems

Problem 2.1. (ES°) Let $ABCD$ be a quadrangle. Prove that $|AB| + |BC| + |CD| > |AD|$. *Hint: triangle inequality*

Problem 2.2. (ES°) In the triangle ABC the angle A is greater than the angle B. Prove that $|BC| \geq \frac{|AB|}{2}$. *Hint: apply both propositions I.18 and I.20*

Problem 2.3. (S°) ABC is an isosceles triangle, where one side is $4°$ and another is $10°$. Find its perimeter.

Problem 2.4. (ES°) Assume ABC is an acute isosceles triangle with apex A. Prove that any segment AD connecting the vertex with the base BC is less than the the leg $|AB|$. *Hint: one of the angles $\angle ADB$ and $\angle ACB$ is at least* $90°$.

Problem 2.5. (ES°) Given two points A, B, and a line l, such that A and B lie on the different sides of L. Find a point C on l, such that the distance $|AC| + |BC|$ is the smallest possible.

Problem 2.6. (S$^\bullet$) Suppose that the sum of two sides $|AB| + |BC|$ in a spherical triangle is less than π. Prove that the sum of the two opposite angles $m\angle C + m\angle A$ is less than $180°$. *Hint: Let D be the point antipodal to A. Apply proposition I.18 to $\triangle DBC$.*

Problem 2.7. (ES$^\bullet$) Take an arbitrary point P inside the triangle ABC. Prove that the sum of distances from P the vertices is greater than half the perimeter. *Hint: sum up three instances of the triangle inequality.*

Problem 2.8. (S$^\bullet$) Suppose that the sum of two sides in a spherical triangle is less than π. Prove that the median to the third side is less than half the sum of the first two. *Hint: recall Euclid's proof of I.16*

Problem 2.9. (ES$^\bullet$) In some country there are several cities, and the distances between all of them are pairwise different. On one fine morning, a plane takes off from each city and lands in the nearest neighboring city. Can more than five planes land in one city? *Hint: Assume at least* 6 *planes landed in city A. Consider their starting cities.*

Problem 2.10. (S$^\bullet$) Does there exist a spherical triangle all of whose sides are at least $\frac{2\pi}{3}$? *Hint: Let D be the point antipodal to A. Apply proposition I.20 to $\triangle DBC$.*

Problem 2.11. (3d$^\bullet$) Can there be three vectors in space such that all pairwise angles between them are greater than $120°$?

Problem 2.12. (S$^\bullet$) Given a spherical triangle that is not isosceles, and the sum of two sides is less than π, draw a median to the third side. Prove that of the two angles which the median forms with the two adjacent sides, the smaller angle is adjacent to the larger side. *Hint: double the median and apply proposition I.18*

Problem 2.13. (ES$^\bullet$) Given two points A, B, and a line l, such that A and B lie on the same side of L. Find a point C on l, such that the distance $|AC| + |BC|$ is the smallest possible. On a sphere, also find a point such that it is the largest possible.

Problem 2.14. (ES$^\bullet$, Euclid's I.24) Let ABC and $A'B'C'$ be two triangles such that $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $m\angle A > m\angle A'$. Prove that $|BC| > |B'C'|$.

Define a *midline* of a triangle ABC to be the line joining the two midpoints M and N of the sides, say, AB and AC, respectively. The side AC is then said to be the side opposite to the midline.

Problem 2.15. (S$^{\bullet\bullet}$) Prove that in any spherical triangle, the midline is greater than half the opposite side. *Hint: Let ABC be the triangle, D be the midpoint of AB and E be the midpoint of AC. Complete ADE to a "parallelogram".*

Problem 2.16. (ES$^{\bullet\bullet}$) Suppose that base BC of a triangle ABC is known, and the sum of the measures of the angles at B and C is known. Prove that regardless of the measures of the angles at B and C, the bisector of the angle at A passes through a fixed point.