Problems

E in front of the problem indicates that the problem should be solved in Euclidean geometry. S indicates that it should be solved in spherical geometry. Circles indicate difficulty.

Problem 1. (ES°) Let the quadrilateral $ABCD$ be such that the diagonals AC and BD share a midpoint M.

a) Draw a picture.

b) To prove that $AB = CD$, we employ a two-column proof:

Statement	Reason
1. $m\angle AMB = m\angle DMC$	1. Vertical angles are equal.
2. $AM = CM$	2. Given.
3. $BM = DM$	3. Given.
4. $\triangle AMB \cong \triangle DMC$	4. SAS (1,2,3)
5. $AB = CD$	5. Corresponding parts of congruent triangles are congruent.

c) Fill in the gaps in the following proof that if $m\angle ABM = m\angle BAM$, then $m\angle BCM = m\angle CBM$.

Statement	Reason
1. $m\angle ABM = m\angle BAM$	1. Given
2. Triangle AMB is isosceles	2. Converse of the Base Angles Theorem
3. $AM = BM$	3. ________________
4. ________________	4. M is the midpoint of AC
5. $BM = CM$	5. ________________
6. ________________	6. Definition of isosceles triangle
7. $m\angle BCM = m\angle CBM$	7. Base Angles Theorem

d) Fill in the gaps in the following proof that if $m\angle BAM = m\angle DAM$, then $m\angle ABM = m\angle CBM$.

Statement	Reason
1. $m\angle AMD = m\angle BMC$	1. ________________
2. ________________	2. M is the midpoint of AC
3. ________________	3. M is the midpoint of BD
4. ________________	4. SAS (1,2,3).
5. $m\angle DAM = m\angle BCM$	5. ________________
6. $m\angle BAM = m\angle DAM$	6. Given.

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Statement	Reason
7. $m\angle BAM = m\angle BCM$	7. Transitivity of Equality.
8. ________________	8. Converse of the Base Angles Theorem.
9. $AB = BC$	9. ________________
10. ________________	10. SAS (2,7,9) *or SSS (2, 3, 9)*
11. $m\angle ABM = m\angle CBM$	11. Corresponding parts of congruent triangles are congruent.

Problem 2. (ES°) In the Figure 1, $m\angle BAO = m\angle DCO$, $m\angle BAC = m\angle DCA$. Prove that $\triangle ABC \cong \triangle CDA$. Use a two-column proof to justify your answer.

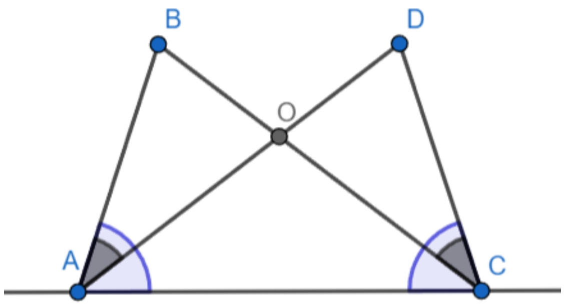

Figure 1: Problem 2

Problem 3. (S°) In the Figure 2, it is known that $m\angle BAC = m\angle ACD$, $m\angle BCA = m\angle CAD$ and $|AB| = \frac{\pi}{6}$ (i.e. 30°) and $|BC| = \frac{\pi}{3}$ (i.e. 60°). Find $|AD|$ and $|CD|$. Use a two-column proof to justify your answer.

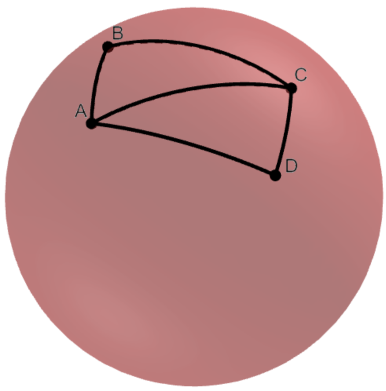

Figure 2: Problem 3

Problem 4. (S°) In an isosceles triangle ABC with base AC, $m\angle BCA = 60°$, $m\angle ABC = 100°$, BD is the median. Draw a picture. Find the angles of the triangle ABD. Use a two-column proof to justify your answer.

Recall that a *perpendicular bisector* of a segment is the line that is perpendicular to the segment and its midpoint.

Problem 5. (ES°) Prove that every point of a perpendicular bisector of a segment is equidistant from its endpoints. Prove, conversely, that every point that is equidistant from the ends of a segment, lies on its perpedicular bisector.

Problem 6. (S°) Prove that two great circles are perpendicular if and only if they pass through each others' poles.

Problem 7. (S°) Prove that for any two distinct great circles there is a unique great circle perpendicular to both of them.

Problem 8. (ES$^\bullet$) In a convex pentagon $ABCDE$, angles B and D are equal, sides AB and ED are equal, and sides BC and CD are equal. Prove that the diagonals AD and BE are equal.

Problem 9. ($^\bullet$) Consider the SAA congruence condition: let two triangles be such that they have two pairs of equal corresponding angles, and a pair of sides opposite to one of the angles is equal. Is it true that the triangles are necessarily congruent a) in plane geometry b) in spherical geometry?

Problem 10. (ES$^\bullet$) A point M inside a triangle ABC is such that the line BM bisects both angles AMC and ABC. Prove that BM is perpendicular to AC.

Recall that a quadrilateral is called a *rhombus*, if all of its four sides are equal.

Problem 11. (ES$^\bullet$)

a) Prove that the opposite angles of a rhombus are equal.

b) Prove that the diagonals of a rhombus are perpendicular.

Problem 12. (E$^\bullet$) Prove that if a median of a triangle bisects its angle, then the triangle is isosceles. Is it true in spherical geometry?

Problem 13. ($^{\bullet\bullet}$)

a) In a plane triangle with $\alpha = 30°, \beta = 60°, \gamma = 90°$, prove that $c = 2a$.

b) In a spherical triangle with $\alpha = 45°, \beta = 45°, \gamma = 120°$, prove that $AB = \frac{\pi}{2}$.

c) In a spherical triangle with $a = \frac{\pi}{4}, b = \frac{\pi}{4}, c = \frac{\pi}{3}$, prove that $\gamma = 90°$.

Problem 14. (S$^{\bullet\bullet}$) Assume that in a spherical right triangle, the sum of a leg and its adjacent acute angle is 90°. Prove that the sum of the second leg and the hypotenuse is 90°.