ORMC AMC 10/12 Group
Week 10: Competition

March 10, 2024

1 Relay Round

1-1 A rectangular box has dimensions $8 \times 10 \times 12$. Compute the fraction of the box's volume that is *not* within 1 unit of any of the box's faces.

1-2 Let T be The Number You Will Receive (TNYWR). Compute the largest real solution x to $(\log(x))^2 - \log(\sqrt{x}) = T$.

1-3 Let T be The Number You Will Receive (TNYWR). Kay has $T + 1$ different colors of fingernail polish. Compute the number of ways that Kay can paint the five fingernails on her left hand by using at least three colors and such that no two consecutive fingernails have the same color.

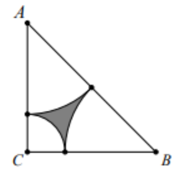
2-1 Compute the number of integers n for which $2^4 < 8^n < 16^3 2$.

2-2 Let $T = TNYWR$. Compute the number of positive integers b such that the number T has exactly two digits when written in base b.

2-3 Let $T = TNYWR$. Triangle ABC has a right angle at C, and $AB = 40$. If $AC - BC = T - 1$, compute $[ABC]$, the area of $\triangle ABC$.

2 Team Round

1. In $\triangle ABC$, $m\angle A = m\angle B = 45°$, and $AB = 16$. Mutually tangent circular arcs are drawn centered at all three vertices; the arcs centered at A and B intersect at the midpoint of $\overline{AB}$. Compute the area of the region inside the triangle and outside of th three arcs.



2. Compute the number of ordered pairs of integers (a, b) such that $1 < a \leq 50$, $1 < b \leq 50$, and $\log_b(a)$ is rational.

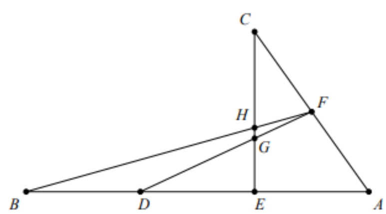
3. Suppose that 5-letter "words" are formed using only the letters A, R, M, and L. Each letter need not be used in a word, but each word must contain at least two distinct letters. Compute the number of such words that use the letter A more than any other letter.

4. Positive integers $a(1), a(2), a(3), \ldots$ form an arithmetic sequence. If $a(1) = 10$ and $a(a(2)) = 100$, compute $a(a(a(3)))$.

5. The graphs of $y = x^2 - |x| - 12$ and $y = |x| - k$ intersect at distinct points A, B, C, and D, in order of increasing x-coordinates. If $AB = BC = CD$, compute k.

6. The zeros of $f(x) = x^6 + 2x^5 + 3x^4 + 5x^3 + 8x^2 + 13x + 21$ are distinct complex numbers. Compute the average value of $A + BC + DEF$ over all possible permutations (A, B, C, D, E, F) of these six numbers.

7. Given noncollinear points A, B, C, segment $\overline{AB}$ is trisected by points D and E, and F is the midpoint of segment $\overline{AC}$. $\overline{DF}$ and $\overline{BF}$ intersect $\overline{CE}$ at G and H, respectively. If $[DEG] = 18$, compute $[FGH]$.



8. The *taxicab distance* between points (x, y, z) and (a, b, c) is given by

$$d((x, y, z), (a, b, c)) = |x - a| + |y - b| + |z - c|.$$

The region $\mathcal{R}$ is obtained by taking the cube $\{(x, y, z) : 0 \leq x, y, z \leq 1\}$ and removing every point whose taxicab distance to any vertex of the cube is less than $\frac{3}{5}$. Compute the volume of $\mathcal{R}$.

9. Let a, b be real numbers such that

$$a^3 - 15a^2 + 20a - 50 = 0 \quad \text{and} \quad 8b^3 = 60b^2 - 290b + 2575 = 0$$

Compute $a + b$.

10. Two square tiles of area 9 are placed with one directly on top of the other. The top tile is then rotated about its center by an acute angle θ. If the area of the overlapping region is 8, compute $\sin(\theta) + \cos(\theta)$.

3 Individual Round

1. Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$

(A) $3^{127}+2^{127}$ **(B)** $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$ **(C)** $3^{128}-2^{128}$ **(D)** $3^{128}+2^{128}$ **(E)** 5^{127}

2. Let f be a function for which $f\left(\frac{x}{3}\right)=x^2+x+1$. Find the sum of all values of z for which $f(3z)=7$.

(A) $-1/3$ (B) $-1/9$ (C) 0 (D) 5/9 (E) 5/3

3. For a certain complex number c, the polynomial

$$P(x)=(x^2-2x+2)(x^2-cx+4)(x^2-4x+8)$$

has exactly 4 distinct roots. What is $|c|$?

(A) 2 **(B)** $\sqrt{6}$ **(C)** $2\sqrt{2}$ **(D)** 3 **(E)** $\sqrt{10}$

4. When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n?

(A) 42 **(B)** 49 **(C)** 56 **(D)** 63 **(E)** 84

5. Let $\overline{AB}$ be a diameter in a circle of radius $5\sqrt{2}$. Let $\overline{CD}$ be a chord in the circle that intersects $\overline{AB}$ at a point E such that $BE=2\sqrt{5}$ and $\angle AEC=45^\circ$. What is CE^2+DE^2?

(A) 96 **(B)** 98 **(C)** $44\sqrt{5}$ **(D)** $70\sqrt{2}$ **(E)** 100

6. The base-nine representation of the number N is $27{,}006{,}000{,}052_{\text{nine}}$. What is the remainder when N is divided by 5?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

7. Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10%?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

8. An integer N is selected at random in the range $1 \leq N \leq 2020$. What is the probability that the remainder when N^{16} is divided by 5 is 1?

(A) $\frac{1}{5}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{3}{5}$ **(D)** $\frac{4}{5}$ **(E)** 1

9. Two cubical dice each have removable numbers 1 through 6. The twelve numbers on the two dice are removed, put into a bag, then drawn one at a time and randomly reattached to the faces of the cubes, one number to each face. The dice are then rolled and the numbers on the two top faces are added. What is the probability that the sum is 7?

(A) $\frac{1}{9}$ (B) $\frac{1}{8}$ (C) $\frac{1}{6}$ (D) $\frac{2}{11}$ (E) $\frac{1}{5}$

10. Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

(A) 9 **(B)** 16 **(C)** 25 **(D)** 36 **(E)** 37