GAMES

MATH CIRCLE (ADVANCED) 11/18/2012

Today we will study games. Unless stated otherwise, there are two players: I and II. Whenever we ask "Who wins?" we are assuming both players use an optimal strategy.

1) a) A rook stands on square $a1$ of a chessboard. Players take turns moving the rook to the right or up. The player who can place the rook on $h8$ wins. Who wins?

The P-positions are the main diagonal $(a1, b2, \dots, h8)$. Thus, Player II wins.

b) A king is placed at $a1$. Players take turns moving the king right, up, or diagonally to the up and right. The player who places the king on $h8$ wins. Who wins?

The P-positions are squares in columns b, d, f, h AND rows $2, 4, 6, 8$.

c) Suppose the piece is placed on a different square than $h8$. List all the squares such that Player II wins the game in a) AND the game in b).

From above it is clear the answer is $b2, d4, f6$.

We divide all the possible positions in a game into **P-positions** and **N-positions.** A P-position is one that is winning for the previous player (that is, the one who just moved) while a N-position is one that is winning for the next player.

Notice in answering the 1), you probably used P and N-positions without even knowing it!

2) Repeat problem 1) with the four center squares of the chessboard $(d4, d5, e4, e5)$ removed. Use P-positions and N-positions in your reasoning!

a) P-positions: $a1, b4, c5, d2, e3, f6, g7, h8$.

b) P-positions: same as problem 1 for rows $5-8$ and columns $e-h$ (of course $e5$ is not a P-position). The others are $b2, a4, c4, d1, d3$.

c) Use a) and b)

3) a) A queen stands on $c1$. Players take turns moving the queen right, up, or diagonally to the up and right. The player who places the queen on $h8$ wins. Who wins?

Hint: P-positions are $a4, d1, c5, e3, f7, g6, h8$.

b) There are two piles of stones, one with 7 and the other with 5. Players alternate taking any number of stones from one of the piles, or an equal number of stones from each pile. The player who cannot move loses. Who wins?

Hint: Show this is exactly the same game as a).

4) Starting with 0, players take turns adding an integer from 1 to 9 to the current number. The player who reaches 100 wins. Who is it?

Hint: P-positions are multiplies of 10.

5) A box contains n matches. Players take turns removing no more than half the matches in the box, but at least one. The player who cannot move loses. For what values of n does Player I win? (As a challenge, prove your answer using induction!)

Hint: P-positions are $2^m - 1$ for $m \geq 1$.

6) There are three piles of stones, containing $50, 60, 70$ stones. A turn consists of dividing each of the piles containing more than one stone into two smaller piles. The player who leaves piles of individual stones wins. Who is it?

Hint: P-positions are those with the largest pile containing $2^m - 1$ stones for $m \geq 1$.

7) A checker for each player is placed at opposite ends of a strip of squares measuring $1 \times n$ for $n \geq 3$. Players take turns moving their checker one or two squares towards the other end. Checkers cannot occupy the same space, but can jump over each other (by moving two squares). The first player to get his square to the other end wins. For what n does Player I win?

Player II wins for $n = 3, 5, 7$ and $n = 4k + 3$ for $k \geq 2$.

8)* There are two piles of matches, one with n and the other with m. Players take turns removing a number of matches from one pile which is one of the divisors of the number of matches in the other pile. The player who removes the last match wins. For what n, m does Player I win?

Hint: P-positions are those with an odd number of matches in each pile.

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"