Intermediate 1 **May 19, 2024**

Probability and Modern Finance

1 Probability and Random Walks

1.1 Review

The probability of an event Y *conditioned* on a possible event X is:

$$P(Y|X) = \frac{P(X \cap Y)}{P(X)}$$

Note: the set intersection symbol ($\cap$) denotes the intersection of events.

Problem 1.1. *Suppose we draw two balls without replacement from an urn with 6 blue balls and 4 red balls. What is the probability we will get two blue balls?*

Problem 1.2. *Suppose we roll a four-sided die then flip that number of coins. What is the probability we will get exactly one Heads?*

We say two events A and B are *independent* if the occurence of one event doe not affect the probability that the other occurs, i.e., $P(B|A) = P(B)$. Alternatively, this can be expressed as $P(A \cap B) = P(A) \cdot P(B)$.

Problem 1.3. *Suppose you roll two dice. Consider the following events:*

$$A = \text{“the first die is 4.”}$$

$$B_1 = \text{“the second die is 2.”}$$
$$B_2 = \text{“the sum of the two dice is 3.”}$$
$$B_3 = \text{“the sum of the two dice is 9.”}$$
$$B_4 = \text{“the sum of the two dice is 7.”}$$

Which of the following pairs of events are independent?

- A *and* B_1
- A *and* B_2
- A *and* B_3
- A *and* B_4

Remember, we can extend the definition of independence to more than two events as follows: A set of events $\mathcal{A} = \{A_1, \ldots, A_n\}$ is *independent* if:

- $P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdot \cdots \cdot P(A_n)$
- Every proper subset of events $S \subsetneq \mathcal{A}$ is independent

1.2 Multiplication Rule and Conditional Independence

Recall that we can rewrite the conditional probability statement as:

$$P(X \cap Y) = P(X) \cdot P(Y|X)$$

This is known as the *multiplication rule* of probability.

Problem 1.4. *Prove the following equation using the multiplication rule:*

$$P(X \cap Y \cap Z) = P(X) \cdot P(Y|X) \cdot P(Z|X \cap Y)$$

Problem 1.5. *Using an inductive argument, prove the following equation:*

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

Problem 1.6. *Three cards are dealt successively at random and without replacement from a standard deck of 52 playing cards. What is the probability of receiving, in order, a king, a queen, and a jack?*

Problem 1.7. *A bag of marbles has 10 green marbles, 20 blue marbles, and 30 red marbles. Drawing one at a time without replacement, what is the probability of drawing two red marbles and then two blue marbles?*

Just like with regular probability, we can define a notion of independence over conditional probability. We say events A and B are *conditionally independent* given a possible event C if the occurrence of one event does not affect the probability that the other occurs *when given the fact that event C has occurred*, i.e., $P(B|A \cap C) = P(B|C)$. Alternatively, this can be expressed as $P(A \cap B|C) = P(A|C) \cdot P(B|C)$.

Problem 1.8. *A box contains two coins: a regular, fair coin and a coin where both sides are heads. Suppose you choose one coin at random and toss it twice. Consider the following events:*

$$A = \text{“the first toss lands Heads.”}$$

$$B = \text{“the second toss lands Heads.”}$$

$$C = \text{“the fair coin was selected.”}$$

Are A and B independent? Are A and B conditionally independent given C?

Problem 1.9. *You have 6 flashcards, each with one of the following words: "broccoli", "carrot", "grape", "peach", "pear" and "potato". You shuffle them and draw 2 at random. Consider the following events:*

$$A = \text{"the third letter on the first card is a vowel."}$$

$$B = \text{"the third letter on the second card is a vowel."}$$

$$C = \text{"both cards have fruits written on them."}$$

Are A and B independent? Are A and B conditionally independent given C?

Problem 1.10. *Is it possible to have events A, B and C such that A and B are independent, but are* ***not*** *conditionally independent given C? Construct such events if so, and explain your reasoning if not.*

1.3 Markov Property and Random Walks

Recall that a *random variable* is a collection of outcomes, each with some assigned value and some assigned probability. For example, a fair coin flip can be thought of as a random variable X with outcomes 0 (Tails) and 1 (Heads), and $P(X = 0) = P(X = 1) = \frac{1}{2}$.

We say that a sequence of random variables $X_0, X_1, X_2, \ldots$ has the *Markov property* if, given the current state X_n, any other information about the past is irrelevant for predicting the next state X_{n+1}. Mathematically, this is equivalent to stating:

$$P(X_{n+1} = k_{n+1}|X_n = k_n, \ldots, X_1 = k_1, X_0 = k_0) = P(X_{n+1} = k_{n+1}|X_n = k_n)$$

We refer to any sequence of random variables that has the Markov property as a *Markov chain.*

Example 1. *Consider one of the random walk games from the in-class activity: a walker starts at position 0 on the number line and, each turn, either moves one step to the right or one step to the left based on the result of an independent, fair coin flip. Let X_i represent the position of our walker on the number line after i coin flips. Does the sequence of random variables $X_0, X_1, X_2, \ldots$ have the Markov property? Well, we notice that the position of our walker after $i+1$ coin flips is completely determined by the walker's position after i flips and the result of the $i+1^{th}$ flip; if we already know where the walker is after the i^{th} flip, knowing where they were at any of the previous steps doesn't give us any additional information! In particular:*

$$P(X_{i+1} = k_i{+}1|X_i = k_i, \ldots, X_1 = k_1, X_0 = k_0) = P(X_{i+1} = k_i{+}1|X_i = k_i) = 0.5$$

$$P(X_{i+1} = k_i{-}1|X_i = k_i, \ldots, X_1 = k_1, X_0 = k_0) = P(X_{i+1} = k_i{-}1|X_i = k_i) = 0.5$$

Therefore, we can conclude that this random walk game satisfies the Markov property.

Problem 1.11. *List at least 3 different examples of Markov chains.*

Problem 1.12. *Let $X_0, X_1, X_2, \dots$ be a Markov chain. Prove that X_{n+1} and X_k are conditionally independent given X_n, for all $k < n$.*

Problem 1.13 (Gambler's ruin)**.** *Consider another of the random walk games from the in-class activity: suppose you enter a casino with $\$k_0$ and each turn you win \$1 with probability p and lose \$1 with probability $1 - p$. Suppose further that you quit playing if your fortune reaches $\$N$, and that the casino kicks you out if your fortune reaches \$0. Once either of these occur, your fortune stays constant for every remaining turn. Show that your fortunes after every turn form a Markov chain.*

1.4 Guaranteed Profit!

Problem 1.14. *A casino near you, The Gamblers Paradise, only hosts a certain gambling game: after taking bets from every player at the table, they flip a coin that turns Heads with probability 1%, and Tails with probability 99%. If the coin turns up Heads, everyone at the table receives a payout equal to their bet amount, and if the coin turns up Tails, everyone loses their bet amount. After consistently losing patrons to casinos that host more traditional games, the manager of The Gambler's Paradise introduces some new policies: patrons can make bets as large as they want and are allowed to go into as much debt as they desire, so long as they pay what they owe before they leave the casino. Can you find a way to take advantage of these new policies and make guaranteed (i.e. with 100% probability) profit?*

2 Modern Finance

2.1 Risk, Risk-Free Assets, Bonds

Recap: Return Measures
Last week, we discussed ”Holding Period Return” as a standardized measure to understand the return from an investment. Recall the formula for Holding Period Yield:

$$HPR = \frac{\text{Ending Investment Value + Income - Beginning Investment Value}}{\text{Beginning Investment Value}} \tag{1}$$

Problem 2.1. *You buy gold for \$2000 per ounce and hold onto it for 3 years. Over this period, the value of gold appreciates to \$5000 per ounce. There are no additional earnings from the gold during the holding period. Calculate the HPR of this gold investment.*

Annualized HPR
A further standardized way to explain a return on investment is to explain it every year. This is the common language of the financial world, where different bundles of investments are compared in terms of their yearly return they are able to generate, even though they are hold for shorter/longer periods. By fixing returns at a timeframe, consistency is achieved. The formula for Annualized HPR is written as:

$$Annualized\ HPR = aHPR = \left(\frac{\text{End. Inv. Value + Income}}{\text{Beginning Investment Value}}\right)^{\frac{1}{t}} - 1 \tag{2}$$

Problem 2.2. *You invest \$200 today and accumulate \$280 after 5 years. What is your aHPR?*

Risk

In finance, risk is defined as the chance that an outcome or an investment's actual gains differ from the expected value of our investment, which we discussed last week. Recall that the expected value of a random variable X with possible outcomes $k_1, k_2, \ldots, k_n$ and respective probabilities $p_1, p_2, \ldots, p_n$ is:

$$E[X] = k_1 \cdot p_1 + k_2 \cdot p_2 + \cdots + k_n \cdot p_n$$

We then define variance to look at the difference between an investment's actual return potential versus the expected (mean) return:

$$Variance = E[(X - E[X])^2] \tag{3}$$

Hence, we have:

$$Standard\ Deviation = \sqrt{Variance} \tag{4}$$

We are going to use standard deviation as our main measure of volatility of asset prices in comparison to their historical averages.

Problem 2.3. *Tossing a fair coin, you bet \$1 on heads. If the coin lands heads up, you win \$2. Otherwise you lose your bet. Find the expected win and the variance.*

Problem 2.4. *A slot machine pays out \$1 one-third of the time, \$2 one-tenth of the time, and a jackpot of \$100 1/500 of the time. Find the expected win and the variance.*

Problem 2.5. *You draw a card from a shuffled standard deck, and are paid out as follows: if you draw an ace, you win \$50; if you draw a face card, you win \$10; otherwise, you win nothing. Find the expected win and the variance.*

Risk-free Asset

A risk free asset is an asset with a certain return, where the standard deviation is 0. Consider putting your money to a bank for a specified interest rate. It means that you are virtually guaranteed to get a specific return in terms of interest, as written on the specifications of the bank account, with a risk of 0. (Excluding the fairly uncommon incidents of financial crises and bank bankruptcies.). This concept will be essential when we move onto the **risk-free rate**, which is denoted by the interest rate of a short-term US Treasury Bill, a financial asset under the class of **Bonds**.

Risk-free assets provide a benchmark in terms of explaining what is an investors' expected rate of return with zero risk. It is usually the lowest level of return that an investor should ideally accept, and turn down any investment opportunity that offers less than the risk-free return rate available on the market.

Risk vs Reward
The risk-return trade-off is the balance between the desire for the lowest possible risk and the highest possible returns.In general, low levels of risk are associated with low potential returns and high levels of risk are associated with high potential returns. (Investopedia)

Problem 2.6. *Consider the table below that outlines two different investment opportunities. Does one of the investment opportunities dominate the other, i.e., provide a higher expected return for lower risk?*

Investment	Volatility	Expected Return
A	2%	%7
B	3%	%5

Table 1: Two Investments

Bonds

Fundamentally, bonds can be thought of as the opposite of taking a loan or having debt. Officially, a bond is an investment product where individuals lend money to a government or company at a certain interest rate for an amount of time. The institutions (governments, corporations, municipalities etc.) issue these bonds to finance their operations and access to investor funds, whereas investors buy bonds in the hope of a certain expected return after.

We can discuss two types of bonds, zero-coupon and coupon bonds, where the main difference is that the zero-coupon bonds pay back with the total sum of interest rate on principal after the specified time period. On the other hand, coupon bonds pay their interest over periodic coupon payments, and pay back the principal at the end. For the simplicity and the scope of this class, we will be sticking with zero-coupon bonds.

The US Treasury Bill:

One of the most widely recognized investment products is the US Treasury Bill, the short-term , zero-coupon U.S. Government debt certificate ranging from 4-week to 1-year duration, also known as maturity. Usually 3-month US T-bill is an industry benchmark on the risk-free rate that we discussed earlier, serving as an effective risk-free investment opportunity to compare risky investments. As of May 17, the 3-month US Treasury bill stands at a rate of 5.25% annually.

Problem 2.7. *You buy a 2-year zero coupon bond for $250. Hold it until maturity, assuming the bond's face value (the final amount you will get after holding into the bond) is $1000. Calculate your aHPR. Think about what does this value mean for bonds.*

Problem 2.8. *Suppose you buy the bond mentioned above again, but sell it after one year when the price of the bond was $400. Calculate your aHPR. Think about what does this value mean for bonds.*

2.2 World of Stocks: Random Walk, Binomial Tree, Simple Call Option, Naive vs Actual Option Pricing

In our financial literacy series, we have so far covered the fundamentals of income-generating investment, the idea of interest and the time value of money, discounting, annuities, perpetuities, and bonds. What many of these concepts have in common is that they are centered around a specific interest rate or a discount rate, as dubbed fixed-income securities. Fixed-income securities make up the largest part of the financial world in terms of transaction size.

This time, we are going to discuss something different, equity securities, also known as stocks. As we discussed last week, stock is a financial asset that represents the fractional ownership of the issuing corporation. Units of stocks are called **"shares"**, and buying a **share** of a company means buying a fraction of the ownership of a company. Stocks are bought and sold in stock markets and are the foundation of many investors' portfolios.

What drives the movement of stock prices is essentially the demand and supply for a specific company's shares. For a simple look, We will assume that stock market consists of a very large number of investors, where no single investor has the power to manipulate markets, and each with equal access to information and investment opportunities. We will also assume that the stock market is efficient, i.e. share price of a company reflects all the information about supply and demand, and stocks are already at their fair market value.

Binomial Stock Probability Tree and Random Walk Theory: A strong argument in the world of finance is that asset prices are essentially random and unpredictable. We know that a stock can either go up or down in price, with varying probabilities. If we treat the stock price as a random variable, we can even come up with a binomial stock price tree that explains the random walk of stock movement more clearly.

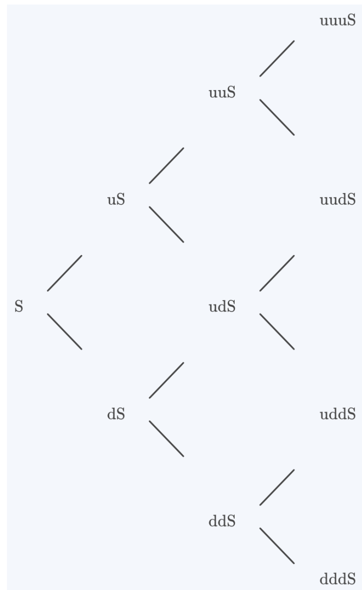

Figure 1: Binomial Stock Price Tree (analystprep.com)

In this stock price tree, S denotes the stock price today. u denotes one plus rate of return when stock price goes up, and d denotes one plus the rate of return when stock price goes down. For example, uS means the stock price is up after one period, while dS means the stock price is down.

Problem 2.9. *Suppose the initial stock price is \$100, $u = 1.5$, $d = 1/1.5$, and the probability of an "up" move is 0.8, calculate the stock prices in each case after one period with probabilities.*

Problem 2.10. *Suppose the initial stock price is \$100, $u = 1.05$, $d = 1/1.05$, and the probability of an "up" move is 0.6, calculate the stock prices in each case after two periods with probabilities.*

While this is a simplified model concerning a single stock, it is essentially a part of the larger scale of the financial world where countless numbers of forecasts of this nature are being conducted to estimate stock prices.