RECURRENCES AND GENERATING FUNCTIONS

GLENN SUN
UCLA MATH CIRCLE ADVANCED 1

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1 Introduction

You may already be familiar with *arithmetic sequences* (like $1, 4, 7, 10, \ldots$, where terms have a common difference) and *geometric sequences* (like $2, 4, 8, 16, \ldots$, where terms have a common ratio). Each of these can be defined using a *recurrence*, meaning one generates the next term based on the previous one(s). For example, $1, 4, 7, 10, \ldots$ is the sequence given by $a(0) = 1$, $a(n) = a(n-1) + 3$. Today, we will study more general classes of sequences formed by recurrence relations, as well as a closely related tool called generating functions.

First, try the following warm-up problem.

Problem 1. How many ways can you tile a 2×10 rectangle with dominoes $\{\square\square, \boxminus\}$? To solve the problem, try the following:

1. First, manually count the number ways you can do this for 2×1, 2×2, 2×3, 2×4, and 2×5 rectangles.

2. Do you recognize this sequence? Make a conjecture and show why it's true.

3. Now answer the original question for a 2×10 rectangle.

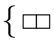
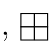
2 Linear recurrences with constant coefficients

Here is the primary object of our study today.

Definition 1. A sequence $a(n)$ is a *linear recurrence with constant coefficients* if it satisfies an equation of the form $a(n) = c_1 a(n-1) + \ldots c_k a(n-k)$, where $c_1, \ldots, c_k \in \mathbb{C}$ (typically we will use $\mathbb{N}$ or $\mathbb{R}$). To fully define such a sequence, we also need to pick *initial conditions* $a(0), \ldots, a(k-1)$.

For example, the Fibonacci sequence you saw in the previous problem, $a(n) = a(n-1) + a(n-2)$ with initial conditions $a(0) = 1$ and $a(1) = 1$, is a linear recurrence with constant coefficients.

Problem 2. Write linear recurrences with constant coefficients for the following problems. Also compute as many initial conditions as you need to begin evaluating the recurrence.

1. $a(n)$ is the number of tilings of a $2 \times n$ rectangle by the tileset $\{$, , $\}$.

2. $a(n)$ is the number of ways to pay for an item costing n dollars using \$1 bills, \$2 bills, and \$5 bills, where the order in which the bills appear matters.

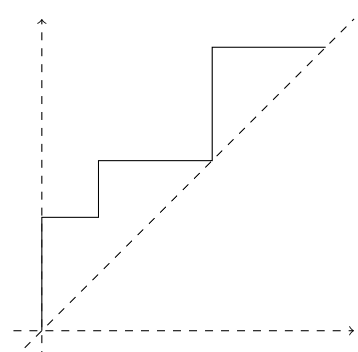
Problem 3. Recurrences that look different can actually define the same sequence. Show that the following recurrences actually define the Fibonacci sequence $a(0) = a(1) = 1$, $a(n) = a(n-1) + a(n-2)$.

1. $a(n) = 1 + \sum_{i=0}^{n-2} a(i)$, with $a(0) = a(1) = 1$.

2. $a(n+1) = \frac{1}{a(n-1)}((-1)^n + a(n)^2)$, with $a(0) = a(1) = 1$.

3 Generating functions

Now, consider the following question: How many paths are there from $(0,0)$ to (n,n), using unit steps up and right only, and stay above (or touch) the line $y = x$? Such paths are called Dyck paths, one for $n = 5$ is illustrated below.



Problem 4. Let $a(n)$ be the number of Dyck paths from $(0,0)$ to (n,n).

1. Compute $a(1)$, $a(2)$, and $a(3)$.

2. Show that this satisfies the recurrence $a(n+1) = \sum_{i=0}^{n} a(i)a(n-i)$. (Hint: consider the dividing Dyck paths into two parts at the first point where the path returns to the line $y = x$.)

The sequence $a(n)$ above is also called the *Catalan numbers* and also count many other problems. See the challenge problems for more.

Part 2 above gave a recurrence for the Catalan numbers, but it it is not a linear recurrence with constant coefficients. We know that different recurrences can give rise to the same sequence, so the question remains: can we find a linear recurrence with constant coefficients for the Catalan numbers?

Spoilers: The answer is no, but how do you explain why? To do this, we will need to introduce the idea of generating functions.

Definition 2. Given a sequence $a(n)$, the (ordinary) generating function of $a(n)$ is the function (in x) given by

$$A(x) = \sum_{n=0}^{\infty} a(n)x^n.$$

(We will not worry about whether or not this infinite sum actually converges.)

Generating functions are an extremely frequently used tool in combinatorics. Some advanced Math Circle worksheets from previous years talk about generating functions before recurrences!

Problem 5. Write the generating function for $a(n) = 1$ (the constant sequence), and then simplify away the big sum.

Problem 6. Show the following:

1. The generating function for $a(n) + b(n)$ is $A(x) + B(x)$.

2. The generating function for $ca(n)$ is $cA(x)$.

3. If $a(0) = 0$, the generating function for $a(n-1)$ is $\frac{1}{x}A(x)$.

Now, we will show you how to simplify the big sum for a more complicated sequence. Let's again take the Fibonacci numbers $a(n) = a(n-1) + a(n-2)$ with $a(0) = a(1) = 1$ for example. It would be nice if we could apply the previous problem, but $a(0) \neq 0$. Instead, we can do the following:

$$\begin{aligned}
A(x) &= a(0) + a(1)x + a(2)x^2 + a(3)x^3 + \dots \\
&= a(0) + a(1)x + (a(1) + a(0))x^2 + (a(2) + a(1))x^3 + \dots \\
&= a(0) + a(1)x + (a(1)x^2 + a(2)x^3 + \dots) + (a(0)x^2 + a(1)x^3 + \dots) \\
&= a(0) + a(1)x + x(A(x) - a(0)) + x^2A(x) \\
&= 1 + x + xA(x) - x + x^2A(x) \\
&= 1 + xA(x) + x^2A(x)
\end{aligned}$$

Now, we can rearrange to solve for $A(x)$ and get that

$$A(x) = \frac{1}{-x^2 - x + 1}.$$

Problem 7. For each of the recurrences you wrote in Problem 2, find and simplify the generating function of the sequence.

In general, we have the following. One direction of the proof is the same algorithm that you have been doing.

Theorem 3. A sequence satisfies a linear recurrence with constant coefficients if and only if its generating function is of the form $\frac{p(x)}{q(x)}$, where p and q are polynomials (such a function is called *rational*).

Problem 8. Let $a(n)$ be the Catalan numbers.

1. Using Problem 4.2, show that the generating function of the Catalan numbers satisfies $A(x)^2 = \frac{A(x)-1}{x}$.

2. Using the previous theorem, conclude that $a(n)$ does not satisfy any linear recurrence with constant coefficients.

4 Extra problems

Problem 9. Show that the Catalan numbers also count:

1. The number of proper expressions using n pairs of parentheses. (For example, ())(is improper while (()) is proper.)

2. The number of full binary trees with $n + 1$ leaves. (In a full binary tree, every internal vertex has two children: a left child and a right child. Ask your instructor if you need more clarification on the definition.)

3. The number of ways you can draw n non-intersecting lines between $2n$ points arranged in a circle.

Problem 10. One reason that linear recurrences with constant coefficients are useful is because we can always get rid of the recursion entirely, and write the nth term of the sequence purely in terms of n. In this problem, you will see how to do this for Fibonacci numbers $a(n)$.

1. We will first look for solutions to the recurrence, without worry about the initial conditions. If one guesses that $a(n) = r^n$ is a solution to $a(n) = a(n-1)+a(n-2)$, what must r be?

2. Let r_1 and r_2 be the two values from part 1. Show that for any two numbers c_1, c_2, $a(n) = c_1 r_1^n + c_2 r_2^n$ also fits the recurrence.

3. Using the initial conditions $a(0) = a(1) = 1$, create and solve a system of equations to find c_1 and c_2.

4. Conclude your final formula for the Fibonacci numbers.

This technique is general and works for all linear recurrences with constant coefficients.

Problem 11. Fix any set of tiles T (each made of one or more 1×1 squares) and a number k. Show that the number of tilings of an $k \times n$ rectangle is always satisfies a linear recurrence with constant coefficients.

Problem 12. Prove the direction of Theorem 3 that we have not yet covered. That is, show that if a sequence has a rational generating function, then it satisfies a linear recurrence with constant coefficients.