Fundamental Groups

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Part 1: Group Presentations

Today, we will be discussing some basics of *algebraic topology*. First, we will study the basics of *group theory*[1]. We will need an important tool called *group presentations* to understand the groups we will be using.
Group presentations have two parts: the alphabet and the rules. Let's look at an example first with just an alphabet:

$$\langle \underbrace{a, b}_{\text{alphabet}} \rangle$$

This group is composed of all words built in the alphabet $\{a, b\}$, with the group operation being concatenation (which we will denote by $\circ$).

So for example, this group has the element aba, and $aba \circ bab = ababab$.
We can also use exponents to denote repeated concatenation: $a^2 = aa, b^3 = bbb$, etc.

In this group, the empty word is the identity. Since writing a blank space can be confusing, we will use e to stand in for the blank word, so $e \circ ab = ab$ not eab.

Further, elements are required to have inverses, so we will allow a^{-1} and b^{-1} to be used in the alphabet as well, with the requirement that any time a sits next to a^{-1} they are both removed.

$$a^{-1}bb^{-1}a \Rightarrow a^{-1} \underbrace{bb^{-1}}_{\text{cancelling } b \text{ and its inverse}} a \Rightarrow \underbrace{a^{-1}a}_{\text{cancelling } a \text{ and its inverse}} \Rightarrow e$$

Now let's add some rules.

$$\langle \underbrace{a, b}_{\text{alphabet}} \mid \underbrace{a^2, b^3}_{\text{rules}} \rangle$$

Here, we say that any time a^2 appears, we remove it (or equivalently, we define $a^2 = e$). Similarly, any time b^3 appears, we remove it.

[1]A group is a set together with a binary operation, with some rules: the operation must be associative, we must have an identity, and every element must have an inverse. We won't need the definition of a group today. If you're curious, you can see why the group presentations actually satisfy these group axioms.

This definition might look strange at first, but we will see where it comes in later when we discuss *loop concatenation*.

Problem 1:
Simplify $ab^5b^{-1}b^{-1}ab^2aa^{-1}ba$.
In general, it can be very hard to actually list all the elements of a group just given its presentation.

Example 2:
Let's look at the group $G = \langle a, b | a^2, b^2, aba^{-1}b^{-1} \rangle$. In this group, let's prove $ab = ba$.

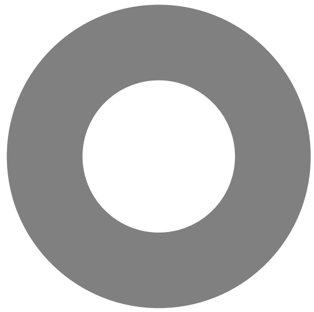
- **A:** We know that $aba^{-1}b^{-1} = e$, as this is a rule we were given in the group presentation.
- **B:** Recall that $(ba)^{-1} = a^{-1}b^{-1}$, so we can write this as $ab(ba)^{-1} = e$.
- **C:** Then concatenating ba onto both sides, we get $ab(ba)^{-1}ba = ba$.
- **D:** But by definition $(ba)^{-1}ba = e$, so $ab = ba$.

Problem 3:
Find all of the elements of G. (Don't forget the identity element!)

Problem 4:
Pick two positive integers p and q. What is the order of $\langle a | a^p, a^q \rangle$?

Part 2: Fundamental Groups

For this packet, we will be handwaving a lot of definitions that rely on *point-set topology*. You will have to use your geometric intuition for a lot of the definitions.
Consider an annulus sitting in $\mathbb{R}^2$, which we will call A. (An annulus is the formal name for the shape of a disc with a hole in it, also known as a washer à la the washer method.)



We would like to answer a very simple question: how many holes does this shape have?

Problem 5:
How many holes does A have?

In order to understand this question, we need to figure out two things: what is a hole, and how do we count holes? For our purposes, a "shape" is simply a subset of $\mathbb{R}^2$.

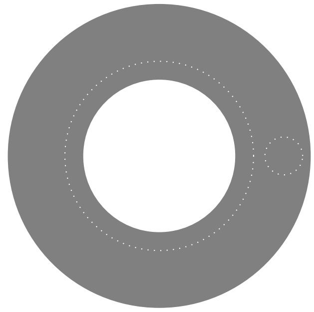
Definition 6:
Given a shape X, a **loop in** X is a curve[2] that starts and ends at the same point, and is a subset of X.

The direction of the loop matters! Going in a circle forwards or backwards count as two different loops!

Problem 7:
Explain why there is a loop that only contains one point.

[2]Technically, a curve is a continuous map $\gamma : [0, 1] \to \mathbb{R}^2$. We don't know what the word "continuous" means, and we can't define it without point-set topology, so we will handwave it away for today and use our intuition.

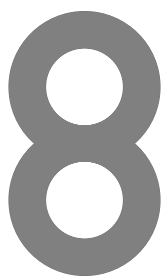
Example 8:
Some loops in A. The direction of the loops is not drawn, but is counter-clockwise.



Definition 9:
Given a shape X, we say two loops in X are **homotopic** if one of them can be *deformed* into the other (while considering direction) without ever leaving X.
What does "deformed" mean? Once again, this is a term that requires some point-set topology to formally define. But we can offer some visual intuition for what a deformation is:
Imagine our loops are pieces of string. We would like to be able to take one piece of string and move it to the other piece of string, without ever leaving our space.

Problem 10:
Are the two loops in Example 8 homotopic? Why or why not?

Problem 11:
In the following shape X, draw two loops that are homotopic and a loop that is not homotopic to either of them. Make sure to mark or note the directions of the loops.

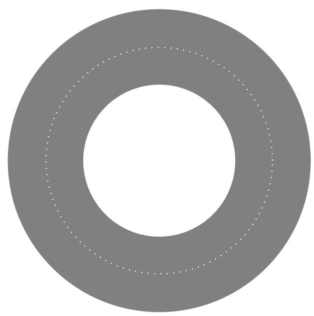


In Problem 7, we determined that one point is a loop.

Definition 12:
We say a loop is **nullhomotopic** if it is homotopic to the loop that only contains one point (the *null* loop).
Intuitively, a loop is nullhomotopic if it can be squished down to a point. In our loop of string analogy, a loop of string is nullhomotopic if it can be “pulled out” without catching on anything.

Problem 13:
Which loops in Example 8 are nullhomotopic? What about the loops you drew in Problem 11?

Problem 14:
Consider the loop below, going counter-clockwise. Is this loop homotopic to the same loop but going clockwise?



We can define a “hole” in a shape X using our definition of nullhomotopic. We would like it to agree with the answer to Problem 5.
A loop is nullhomotopic if there is nothing “stopping” it from being squished. What kind of things can stop it? Only a hole.
But there can be several different loops, even ones that are not homotopic to each other, that are all stopped from being squished because of the same hole.

It turns out that the “number” of holes is a tricky question! To start, we can try to understand the holes themselves and their relations.

Definition 15:
Given a space X and a point $x \in X$, we define the **loop set at** x to be the set of all loops in X that start and end at x. We denote this $L(X, x)$.

Definition 16:
Given a space X and a point $x \in X$, we define the **null-loop set at** x to be the set of all nullhomotopic loops in X that start and end at x. We denote this $N(X, x)$.

Problem 17:
Is $N(X, x) \subset L(X, x)$?

Both $L(X, x)$ and $N(X, x)$ have a lot of redundant information. We'd like to make a group eventually, and we'd like to remove the unnecessary fluff. Let's reduce this information.

First, we would like to understand how to combine loops. If $\ell, \ell' \in L(X, x)$, we can concatenate them to get a new loop $\rho = \ell \circ \ell'$. ρ is simply the loop given by going around ℓ and then around ℓ'.

Second, we would like to be able to invert loops. If $\ell \in L(X, x)$, then we define loop ℓ^{-1} to be ℓ but in the opposite direction.

Problem 18:
We would want $\ell \circ \ell^{-1}$ to be zero (the null loop) if we actually have a group structure. Is this true?

However, it is true that $\ell \circ \ell^{-1}$ is nullhomotopic. That is, $\ell \circ \ell^{-1} \in N(X, x)$.

Problem 19:
Consider four loops $\ell_1, \ell_2, \ell'_1, \ell'_2 \in L(X, x)$, such that ℓ_1 is homotopic to ℓ'_1 and ℓ_2 is homotopic to ℓ'_2. Is $\ell_1 \circ \ell_2$ homotopic to $\ell'_1 \circ \ell'_2$?

It is also true that for $\ell, \ell' \in L(X,x)$ that are homotopic, ℓ^{-1} is homotopic to ℓ'^{-1}.
What we've done so far tells us something interesting: all the information of $L(X,x)$ is *up to homotopy*. What does this mean?

Given two loops $\ell, \ell' \in L(X,x)$ that are homotopic, we gain no information by considering both ℓ and ℓ'. The group structure (concatenation, inversion) doesn't differentiate between two loops that are homotopic.

Problem 20:
Show that for $\ell, \ell' \in L(X,x)$, we have ℓ is homotopic to ℓ' if and only if $\ell \circ \ell'^{-1}$ is nullhomotopic.

Definition 21:
Given a space X and $x \in X$, we define the **fundamental group at** x of X, denoted $\pi_1(X,x)$, as the group with alphabet $L(X,x)$ and rules $N(X,x)$. That is, $\pi_1(X,x) = \langle L(X,x) \,|\, N(x,x) \rangle$.

Another way to define $\pi_1(X,x)$ is $\pi_1(X,x) := L(X,x)$ "up to homotopy". That is, $\pi_1(X,x)$ as a set is given by loops in X that start and end at x, where we treat two loops as the same if they are homotopic. The identity of $\pi_1(X,x)$ is the null loop, and the group operation is concatenation of loops.

When a space X is *path-connected*, then the choice of x does not matter. That is, $\pi_1(X,x) = \pi_1(X,x')$ for any $x, x' \in X$. In this case, we will just write $\pi_1(X)$. All the spaces we consider today will be path-connected, and we won't define this term.

Example 22:
Let's find $\pi_1(A)$ (remember A is our annulus).
We found in Problem 10 that the loop ℓ around the centre of A is not nullhomotopic. In fact, every loop in A that is not nullhomotopic is homotopic to ℓ (why?).

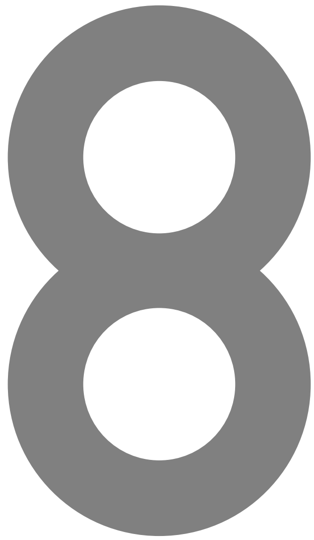
It turns out that $\ell \circ \ell$ is not homotopic to ℓ (why?), and similarly ℓ is not homotopic to ℓ^{-1}. Thus, all the non-nullhomotopic loops are $\ell, \ell \circ \ell, \ell \circ \ell \circ \ell, \ldots, \ell^{-1}, \ell^{-1} \circ \ell^{-1}, \ldots$.
Thus,

$$\pi_1(A) = \mathbb{Z} = \langle \ell \rangle$$

Where instead of writing $\ell \circ \cdots \circ \ell$ we just write the number of ℓs we have. Negative numbers are sums of ℓ^{-1}, and 0 is the null loop. This is the same group as $\langle \ell \rangle$, the group generated by ℓ with no rules.

Why are there no rules? Because there is no combination of ℓ that results in a nullhomotopic loop. There is one generator (every nullhomotopic loop is a power of it or its inverse), ℓ, and no way to make it into a nullhomotopic loop. Thus, $\pi_1(A) = \langle \ell \rangle$.

Problem 23:
Find $\pi_1(X)$ where X is the following shape. *Hint: first, find how many generators there are. Then, try to figure out what the rules should be, if there are any.*



Problem 24:
If we define an annulus to be a 1-holed object, and the above surface to be a 2-holed object, draw a 3-holed object. What is the fundamental group of this object?

This suggests a "definition" of holes, at least for a certain type of shapes.

Problem 25:
Find a link between a n-holed object and its fundamental group. *Hint: think about the group presentation of the fundamental group. How many generators and how many rules does it have?*

Problem 26:
Find $\pi_1(X)$ where X is a torus. $\pi_1(X)$ *can be defined in the exact same way we have done so far, since the torus is a surface. The only difference is it isn't a subset of* $\mathbb{R}^2$.

Part 3: Extremely Challenging Problems

Problem 27:
We define the *bracket* of two spaces X and Y to be $[X, Y] := \{f : X \to Y\}/\sim$, where $f \sim g$ if and only if f is homotopic to g. Thus, $[X, Y]$ is the set of continuous functions from X to Y up to homotopy. Prove that $\pi_1(X) = [S^1, X]$.

Problem 28:
Given a space X, we define the k-th *homotopy group* to be $\pi_k(X) := [S^k, X]$. Then, we see that $\pi_1(X)$ is the 1st homotopy group of X. For $X = S^1$ and $X = S^2$, find $\pi_1(X), \pi_2(X), \pi_3(X)$.

Problem 29:
Show that $\pi_i(S^j) = 0$ for $i < j$.

Problem 30:
Show that $\pi_n(S^n) = \mathbb{Z}$.

Problem 31:
Given a space X, we define the k-th *cohomotopy group* to be $\pi^k(X) := [X, S^k]$. Prove that $\pi_k(S^n) = \pi^n(S^k)$, and find $\pi^2(S^1)$.