NIM

MATH CIRCLE (ADVANCED) 12/02/2012

Recall from last time the idea of dividing a game into **P-positions** and **N-positions.**

A P-position is one that is winning for the previous player (that is, the one who just moved) while a N-position is one that is winning for the next player.

Therefore, if a game starts in a P-position, Player II wins, while if the game starts in a N-position, Player I wins.

The game of Nim: Suppose we have k piles of stones, with n_1 stones in the first pile, n_2 stones in the second pile, etc. Player I and Player II take turns removing any number (at least one!) stone from a single pile. The player who takes the last stone wins.

We will write the above game as $(n_1, n_2, \ldots, n_k)-$Nim.

0) a) List the P-positions and N-positions for Nim played with only one pile ($k = 1$).

b) For what values of n_1 does Player I win?

1) a) Play Nim with piles of size 6 and 4 (i.e. $(6, 4)-$Nim). Who wins?

b) Explain the winning strategy.

2) a) Characterize the P-positions and N-positions for $(n_1, n_2)-$Nim.

b) For what pairs (n_1, n_2) does Player I win?

The Nim-sum (denoted by $\oplus$) of two numbers is computed as follows. Convert the numbers to binary, then add (in binary) WITHOUT carrying any digits, and finally convert back to decimal form. For example,

$$60 \oplus 47 \to \frac{\begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{array}}{\begin{array}{cccccc} 0 & 1 & 0 & 0 & 1 & 1 \end{array}} \to 10011_2 = 19.$$

3) Compute the following Nim-sums:

a) $6 \oplus 4$

b) $12 \oplus 12$

c) $25 \oplus 8$

d) $3 \oplus 4 \oplus 5$

e) $19 \oplus 60 \oplus 47$

f) $2 \oplus 2 \oplus 2 \oplus 2$

g) $5 \oplus 2 \oplus 2 \oplus 2$

4) Write out P-positions and N-positions for $(n_1, n_2, n_3)-$Nim for $n_1+n_2+n_3 \leq 6$. Think of ways to minimize the number of positions you need to write out!

5) a) Prove that $a \oplus b = 0$ if and only if $a = b$.

b) Restate 2) in terms of Nim-sums.

6) a) Characterize the P-positions and N-positions for $(n_1, n_2, n_3)-$Nim.

b) For what triples (n_1, n_2, n_3) does Player I win?

7)* Prove 6)a).

8)* Characterize the P-positions and N-positions for $(n_1, n_2, \ldots, n_k)-$Nim. Prove your answer!

∞) Solve the following games (i.e. try to characterize the P-positions and N-positions and try to prove your answer. However, this may not be possible!).

a) l-Subtraction Game: Similar to $(n_1, n_2, \ldots, n_k)-$Nim, except players can remove at most l stones.

b) Greedy Nim: Similar to $(n_1, n_2, \ldots, n_k)-$Nim, except players can only remove stones from the largest pile.

c) Multiple Pile Nim: Similar to $(n_1, n_2, \ldots, n_k)-$Nim, except players can also remove the same number of stones from both piles.

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"