ORMC AMC 10/12 Group
Week 7: Combinatorics

May 12, 2024

This worksheet will only include exercises, since we have already gone over all of the essential theorems and strategies for solving AMC 10/12 combinatorics and probability problems. As usual, the problems are not isolated in combinatorics, and you'll have to incorporate things you know from algebra, number theory, geometry, etc. Be sure to take your time before diving in to a problem, to think about potential ways to reframe or visualize it. This helps make your work easier, shorter, and less error-prone.

1 Exercises

1. **(1950 AHSME #45)** How many diagonals can be drawn in a polygon of 100 sides?

2. **(1973 AHSME #2)** One thousand unit cubes are fastened together to form a large cube with edge length 10 units; this is painted and then separated into the original cubes. What is the number of these unit cubes which have at least one face painted?

3. **(1974 AHSME #24)** A fair die is rolled six times. What is the probability of rolling at least a five at least five times?

4. **(1985 AJHSME #15)** How many whole numbers between 100 and 400 contain the digit 2?

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5. **(1990 AHSME #9)** Each edge of a cube is colored either red or black. Every face of the cube has at least one black edge. What is the smallest number possible of black edges?

6. **(1974 AHSME #3)** What is the coefficient of x^7 in the polynomial expansion of $(1 + 2x - x^2)^4$?

7. **(2010 ARML I-1)** Compute the number of positive integers less than 25 that cannot be written as the difference of two squares of integers.

8. **(2014 ARML I-2)** Let A, B, and C be randomly chosen (not necessarily distinct) integers between 0 and 4 inclusive. Pat and Chris compute the value of $A+B \cdot C$ by two different methods. Pat follows the proper order of operations, computing $A + (B \cdot C)$. Chris ignores order of operations, choosing instead to compute $(A + B) \cdot C$. Compute the probability that Pat and Chris get the same answer.

9. **(2009 ARML I-3)** Some students in a gym class are wearing blue jerseys, and the rest are wearing red jerseys. There are exactly 25 ways to pick a team of three players that includes at least one player wearing each color. Compute the number of students in the class.

10. **(2009 ARML R1-3)** Kay has 11 different colors of fingernail polish. Compute the number of ways that Kay can paint the five fingernails on her left hand by using at least 3 colors, and such that no two consecutive fingernails have the same color.

11. **(1990 AJHSME #12)** There are twenty-four 4-digit numbers that use each of the four digits 2, 4, 5, and 7 exactly once. When they are listed in numerical order from smallest to largest, what is the number in the 17th position in the list?

12. **(2009 ARML T-3)** The numbers $1, 2, \ldots, 8$ are placed in a 3×3 grid, leaving exactly one blank square. Such a placement is called okay if in every pair of adjacent squares, either one square is blank or the difference between the two numbers is at most 2 (two squares are considered adjacent if they share a common side). If , rotations, etc. of placements are considered distinct, compute the number of distinct okay placements.

13. **(2007 AMC 10A #11)** The numbers from 1 to 8 are placed at the vertices of a cube in such a manner that the sum of the four numbers on each face is the same. What is this common sum?

14. **(2002 AMC 12B #10)** How many different integers can be expressed as the sum of three distinct members of the set $\{1, 4, 7, 10, 13, 16, 19\}$?

15. **(2014 ARML I-5)** The sequence of words an is defined as follows: $a_1 = X$, $a_2 = O$, and for $n \geq 3$, a_n is a_{n-1} followed by the reverse of a_{n-2}. For example, $a_3 = OX$, $a_4 = OXO$, $a_5 = OXOXO$, and $a_6 = OXOXOOXO$. Compute the number of palindromes in the first 1000 terms of this sequence.

16. **(1963 AHSME #27)** Six straight lines are drawn in a plane with no two parallel and no three concurrent. How many regions do they divide the plane into?

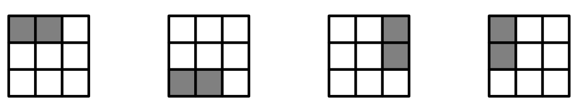
17. **(2009 ARML I-7)** Let A and B be digits from the set $\{0, 1, 2, ..., 9\}$. Let r be the two-digit integer AB and let s be the two-digit integer BA so that r and s are members of the set $\{00, 01, ..., 99\}$. Compute the number of ordered pairs (A, B) such that $|r - s| = k^2$ for some integer k.

18. **(2016 AMC 10A #20)** For some particular value of N, when $(a+b+c+d+1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables a, b, c, and d, each to some positive power. What is N?

19. **(1987 AJHSME #25)** Ten balls numbered 1 to 10 are in a jar. Jack reaches into the jar and randomly removes one of the balls. Then Jill reaches into the jar and randomly removes a different ball. What is the probability that the sum of the two numbers on the balls removed is even?

20. **(2012 ARML I-9)** One face of a $2 \times 2 \times 2$ cube is painted (not the entire cube), and the cube is cut into eight $1 \times 1 \times 1$ cubes. The small cubes are reassembled randomly into a $2 \times 2 \times 2$ cube. Compute the probability that no paint is showing.

21. **(1990 AJHSME #25)** How many different patterns can be made by shading exactly two of the nine squares? Patterns that can be matched by flips and/or turns are not considered different. For example, the patterns shown below are not considered different.



22. **(2009 ARML Local T-6)** Compute the number of values of x with $0 \leq x \leq 2\pi$ for which

$$|\sin(x)| + |\cos(x)| = \frac{4}{3}$$

23. **(2015 AMC 10A #22)** Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

24. **(2003 AMC 12B #19)** Let S be the set of permutations of the sequence $1, 2, 3, 4, 5$ for which the first term is not 1. A permutation is chosen randomly from S. The probability that the second term is 2, in lowest terms, is a/b. What is $a + b$?

25. **(2009 ARML Local T-7)** You are playing a game. Your opponent has distributed five red balls into five boxes randomly. All arrangements are equally likely; that is, the left-to-right placements $[0, 0, 5, 0, 0]$, $[0, 2, 0, 3, 0]$, and $[1, 1, 1, 1, 1]$ are equally likely. You place one blue ball into each box. The player with the most balls in a box wins the box (neither player wins a box with the same number of balls of each color). Whoever wins the most boxes wins the game. Compute the probability you win the game.

26. **(2003 AMC 12A #20)** How many 15-letter arrangements of 5 A's, 5 B's, and 5 C's have no A's in the first 5 letters, no B's in the next 5 letters, and no C's in the last 5 letters?

$$\text{(A) } \sum_{k=0}^{5} \binom{5}{k}^3 \qquad \text{(B) } 3^5 \cdot 2^5 \qquad \text{(C) } 2^{15} \qquad \text{(D) } \frac{15!}{(5!)^3} \qquad \text{(E) } 3^{15}$$

27. **(2004 AMC 12A #20)** Select numbers a and b between 0 and 1 independently and at random, and let c be their sum. Let A, B and C be the results when a, b and c, respectively, are rounded to the nearest integer. What is the probability that $A + B = C$?

28. **(2004 AMC 12B #20)** Each face of a cube is painted either red or blue, each with probability $1/2$. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

29. **(2011 AIME II #7)** Ed has five identical green marbles, and a large supply of identical red marbles. He arranges the green marbles and some of the red ones in a row and finds that the number of marbles whose right hand neighbor is the same color as themselves is equal to the number of marbles whose right hand neighbor is the other color. An example of such an arrangement is GGRRRGGRG. Let m be the maximum number of red marbles for which such an arrangement is possible, and let N be the number of ways he can arrange the $m+5$ marbles to satisfy the requirement. Find the remainder when N is divided by 1000.

30. **(2007 AIME II #13)** A triangular array of squares has one square in the first row, two in the second, and in general, k squares in the kth row for $1 \leq k \leq 11$. With the exception of the bottom row, each square rests on two squares in the row immediately below (illustrated in the given diagram). In each square of the eleventh row, a 0 or a 1 is placed. Numbers are then placed into the other squares, with the entry for each square being the sum of the entries in the two squares below it. For how many initial distributions of 0's and 1's in the bottom row is the number in the top square a multiple of 3?