Advanced Pigeonhole Principle

ORMC

05/12/24

1 Old ORMC Handout Problems

Problem 1.1. You are given 224 integers. Prove that there exist at least two of them such that their difference is divisible by 223.

Problem 1.2. Given 7 lines on the plane, prove that two of them form an angle less than $26°$.

Problem 1.3. There are $n > 1$ people at a party. Prove that among them there are at least two people who have the same number of acquaintances at the gathering. (We assume that if A knows B, then B also knows A.)

Problem 1.4. Is it possible to cover an equilateral triangle with two smaller equilateral triangles? Why or why not?

Problem 1.5. Let n be an integer not divisible by 2 and 5. Show that n has a multiple consisting entirely of ones.

Problem 1.6. Among any five points with integer coordinates in the plane, there exist two such that the center of the line segment connecting them has integer coordinates as well.

Problem 1.7. All the points in the plane are painted with either one of two colors. Prove that there exist two points in the plane that have the same color and are located exactly one foot away from each other.

2 Advanced Pigeonholing

Problem 2.1. A convex set in $\mathbb{R}^n$ contains at least $2^n + 1$ integer points. Prove that at least 3 of them lie on the same line.

Problem 2.2. All the points in the plane are painted with either one of three colors. Prove that there exist two points in the plane that have the same color and are located exactly one foot away from each other.

This problem with the number of colors equal to 4 was solved in 2018 with computer assistance.

Problem 2.3 (Manhattan Mathematical Olympiad 2004)**.** Seven line segments, with lengths no greater than 10 inches and no shorter than 1 inch, are given. Show that one can choose three of them to represent the sides of a triangle.

Problem 2.4 (Canadian Mathematical Olympiad 2004)**.** Let T be the set of all positive integer divisors of 2004^{100}. What is the largest possible number of elements that a subset S of T can have if no element of S is an integer multiple of any other element of S?

Problem 2.5 (IMO 1974)**.** Let M be an arbitrary set of 10 positive integers not greater than 100. Show that there are two disjoint (i.e. having no common elements) subsets of M which have the same sum of their elements.