AMC 8 Training: More Piecewise

April 28, 2024

1 Warm Up

You are buying t-shirts for the math club members. The t-shirt company will charge you according to the function below, where $c(x)$ is the total coast of the shirts and x is the number of t-shirts that you order.

$$\begin{cases} 12x & 0 < x \leq 10 \\ 10x & 11 \leq x \leq 50 \\ 8x & x > 50 \end{cases}$$

a) If you can get 51 members of the math club, how much will the 51 t-shirts cost?
b) If you only have 5 members (sad!), how much will EACH t-shirt cost?
c) You end up with 30 members who want to buy a t-shirt. You want to sell the shirts and end up making \$100 profit for an end-of-year party. How much should you sell each shirt for?

2 Practice!

1) Consider functions f that satisfy

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y|$$

for all real numbers x and y. Of all such functions that also satisfy the equation $f(300) = f(900)$, what is the greatest possible value of

$$f(f(800)) - f(f(400))?$$

(A) 25 **(B)** 50 **(C)** 100 **(D)** 150 **(E)** 200

2) Find the number of integers c such that the equation

$$\big||20|x| - x^2| - c\big| = 21$$

has 12 distinct real solutions.

3) Alice, Bob, and Carol play a game in which each of them chooses a real number between 0 and 1. The winner of the game is the one whose number is between the numbers chosen by the other two players. Alice announces that she will choose her number uniformly at random from all the numbers between 0 and 1, and Bob announces that he will choose his number uniformly at random from all the numbers between $\frac{1}{2}$ and $\frac{2}{3}$. Armed with this information, what number should Carol choose to maximize her chance of winning?

(A) $\frac{1}{2}$ **(B)** $\frac{13}{24}$ **(C)** $\frac{7}{12}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{2}{3}$

4) How many positive integers n satisfy

$$\frac{n + 1000}{70} = \lfloor \sqrt{n} \rfloor?$$

(Recall that $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

(A) 2 **(B)** 4 **(C)** 6 **(D)** 30 **(E)** 32

5) What is the hundreds digit of 2011^{2011}?

(A) 1 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 9

6) The expression

$$(x+y+z)^{2006}+(x-y-z)^{2006}$$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

(A) 6018 (B) $671,676$ (C) $1,007,514$ (D) $1,008,016$ (E) $2,015,028$

7) What is the minimum value of $f(x)=|x-1|+|2x-1|+|3x-1|+\cdots+|119x-1|$?

(A) 49 **(B)** 50 **(C)** 51 **(D)** 52 **(E)** 53

8) Let $[x]$ denote the largest integer not exceeding x. For example, $[2.1]=2$, $[4]=4$ and $[5.7]=5$. How many positive integers n satisfy the equation $\left[\frac{n}{5}\right]=\frac{n}{6}$?

9) Find the integer n satisfying $\left[\frac{n}{1!}\right]+\left[\frac{n}{2!}\right]+...+\left[\frac{n}{10!}\right]=1999$. Here $[x]$ denotes the greatest integer less than or equal to x?

10) What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$$

11) If x is a positive real number, and n is a positive integer, prove that

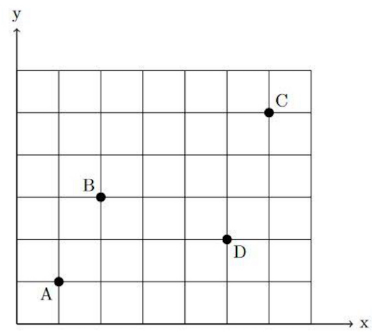
$$[nx] \geq \frac{[x]}{1} + \frac{[2x]}{2} + \frac{[3x]}{3} + ... + \frac{[nx]}{n},$$

where $[t]$ denotes the greatest integer less than or equal to t.

12) Let $[x]$ denote the integer part of x, i.e., the greatest integer not exceeding x. If n is a positive integer, express as a simple function of n the sum

$$\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + ... + \left[\frac{n+2^k}{2^{k+1}}\right] + \dots$$

13) There are four colleges, $A = (1, 1)$, $B = (2, 3)$, $C = (6, 5)$, and $D = (5, 2)$ in the borough of Ideal Manhattan. Please see the picture below. The Burger General Company want to build a cafe G at an intersection of a street and avenue so that $d_1(G, A)+d_1(G, B)+d_1(G, C)+d_1(G+D)$ is the shortest possible. Find all such locations.



14) How many of the first 1000 positive integers can be expressed in the form

$\lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor$,

where x is a real number, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z?

15) Find the number of functions f from $\{0, 1, 2, 3, 4, 5, 6\}$ to the integers such that $f(0) = 0$, $f(6) = 12$, and

$$|x - y| \leq |f(x) - f(y)| \leq 3|x - y|$$

for all x and y in $\{0, 1, 2, 3, 4, 5, 6\}$.

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