ORMC AMC Group: Week 9
Piecewise Functions

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1 Binomial Theorem (Review)

Consider the algebraic expansion of the term $(x+y)^n$. The first few terms are:

1. $(x+y)^2 = x^2 + 2xy + y^2$
2. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
3. $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Notice that the coefficients in the expansion of $(x+y)^n$ form the n-th row of Pascal's triangle. Consider the product of n copies of $(x+y)$:

$$(x+y)^n = \underbrace{(x+y)(x+y)\dots(x+y)}_{n \text{ copies}}$$

The coefficient of $x^k y^{n-k}$ is equivalent to choosing k of the n terms from which to pick x. By definition, there are $\binom{n}{k}$ such subsets. Thus, the binomial theorem states that

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n \\ &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.\end{aligned}$$

1.1 Examples

1. Expand the expression $(1-x)^5$.

2. Find the coefficient of x^8 in the expansion $(x+5)^{10}$.

3. Evaluate $1001^3 - 3 \cdot 1001^2 + 3 \cdot 1001 - 1$.

1.2 Problems

1. Find the value of

$$\binom{6}{1}2^1 + \binom{6}{2}2^2 + \binom{6}{3}2^3 + \binom{6}{4}2^4 + \binom{6}{5}2^5 + \binom{6}{6}2^6.$$

2. Find the remainder when 65^{20} is divided by 512.

3. What is the hundreds digit of 2011^{2011}?

4. Find the units digit of the expansion of

$$(15 + \sqrt{220})^{19} + (15 + \sqrt{220})^{82}.$$

5. (Hard) Evaluate

$$\sum_{n=1}^{\infty} \frac{n^5}{n!}.$$

2 Piecewise Functions

A *piecewise function* is a special type of function which divides up its domain (set of inputs) into subsets, or sub-intervals, and applies different "*subfunctions*" to each of these sets. From the perspective of defining functions, piecewise functions give us a lot more flexibility, since the function we want may behave very differently on different subsets of the domain.

However, we are solving problems, piecewise functions can often be more tedious since there generally isn't a way to deal with the entire function at once. As you will see, the best way to deal with piecewise functions is to deal with the different intervals separately, then put them together at the end to get the final answer.

2.1 Examples

1. What real value of c will make the following function continuous (i.e. there are no "jumps")?

$$f(x) = \begin{cases} x^2 + 2c & \text{if } x \leq 3, \\ 2cx + 1 & \text{if } x > 3. \end{cases}$$

2. The function f is defined on the set of integers and satisfies $f(n) = \begin{cases} n - 3 & \text{if } n \geq 1000 \\ f(f(n+5)) & \text{if } n < 1000 \end{cases}$

   Find $f(84)$.

2.2 Problems

1. Let

$$f(x) = \begin{cases} -x + 3 & \text{if } x \leq 0, \\ 2x - 5 & \text{if } x > 0. \end{cases}$$

   How many solutions does the equation $f(f(x)) = 4$ have?

2. Find the inverse of $f(x)$ if

$$f(x) = \begin{cases} \sqrt{2 - x} & \text{if } x < 0, \\ 1 - x^2 & \text{if } x \geq 0. \end{cases}$$

3. Let $S(x) = 1$ if $x \geq 0$ and $S(x) = 0$ if $x < 0$. Express the function

$$g(x) = \begin{cases} 2 & \text{if} - 3 \leq x \leq 3, \\ 0 & \text{otherwise}, \end{cases}$$

   in terms of $S(x)$ without using piecewise notation.

3 Absolute Value

As you probably know, the absolute value function, generally written $|x|$, denotes the "magnitude" of x. In particular, for the real numbers, it is a piecewise function with the following definition:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Notice that there is an inherent symmetry to absolute value. For example, $x = -1$ is treated the same as $x = 1$. This symmetry is often something you can use to your advantage– if you can find the symmetry, it can cut the amount of work you have to do in half (sometimes even better).

If you're having trouble figuring out an absolute value problem, it often helps to write it out as a piecewise function, similar to above. At the very least, this can help you break it down and get started towards solving it.

3.1 Examples

1. Let a, b be real numbers. Find all x in terms of a and b such that $|x - a| = |x - b|$.

2. Find $f^{-1}(x)$ if $f(x) = x|x| + 2$.

3. Find the area in the x-y plane corresponding to the inequality $(|x| - \frac{1}{2})^2 + (|y| - \frac{1}{2})^2 \leq 1$.

3.2 Problems

1. Find the area of the region enclosed by the graph
$$|x - 60| + |y| = \left|\frac{x}{4}\right|.$$

2. Given that a, b, c are real numbers, find all possible values of the expression
$$\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{abc}{|abc|}.$$

3. What is the product of all the roots of the equation
$$\sqrt{5|x| + 8} = \sqrt{x^2 - 16}.$$

4. Find the number of integers c such that the equation
$$\left||20|x| - x^2| - c\right| = 21$$
has 12 distinct real solutions.

4 The Floor Function

The *floor function*, often written $\lfloor x \rfloor$ denotes the largest integer which is less than or equal to x. For example: $\lfloor 2 \rfloor = 2$, $\lfloor \frac{3}{2} \rfloor = 1$, $\lfloor -5.8 \rfloor = -6$.

A couple corresponding functions are:

- The ceiling function $\lceil x \rceil$ (called "ceil"), which denotes the smallest integer greater than or equal to x. For example: $\lceil 2 \rceil = 2$, $\lceil \frac{3}{2} \rceil = 2$, $\lceil -5.8 \rceil = -5$.
- The fractional-part function $\{x\}$ (called "frac"), which is the remaining value between x and its floor. Specifically, $\{x\} := x - \lfloor x \rfloor$. For example: $\{2\} = 0$, $\{\frac{3}{2}\} = \frac{1}{2}$, $\{-5.8\} = 0.2$.
  Note that $0 \leq \{x\} < 1$ for all real numbers.

When we say $\lfloor x \rfloor = m$, what we are really saying is that $m \leq x < m + 1$. Floor naturally provides a range, which can be used to narrow down on the value you are looking for when solving a problem. A couple other useful tips are:

1. Replace "$\lfloor \cdots \rfloor$" with a single variable like M and solve your equation in terms of an integer first. Then, you can come back and deal with whatever was in the floor function after solving for M.
2. Note that $\lfloor x \rfloor$ behaves approximately the same as x. So, it usually helps to start by removing the "floors" to get a starting approximation/range that you can work from.

4.1 Examples

1. How many positive integers n satisfy the equation $\left\lfloor \frac{n}{5} \right\rfloor = \frac{n}{6}$?

2. Find the integer n satisfying

$$\left\lfloor \frac{n}{1!} \right\rfloor + \left\lfloor \frac{n}{2!} \right\rfloor + \cdots + \left\lfloor \frac{n}{10!} \right\rfloor = 1999.$$

3. Let n an integer. Prove that

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n+1}{2} \right\rceil = n.$$

4.2 Problems

1. Evaluate

$$\left\lfloor \frac{2007^3}{2005 \cdot 2006} - \frac{2005^3}{2006 \cdot 2007} \right\rfloor$$

   without a calculator.

2. How many positive integers n satisfy

$$\frac{n + 1000}{70} = \lfloor \sqrt{n} \rfloor?$$

3. Suppose r is a real number for which

$$\left\lfloor r + \frac{19}{100} \right\rfloor + \left\lfloor r + \frac{20}{100} \right\rfloor + \left\lfloor r + \frac{21}{100} \right\rfloor + \cdots + \left\lfloor r + \frac{91}{100} \right\rfloor = 546.$$

Find $\lfloor 100r \rfloor$.

4. Find

$$\sum_{k=1}^{99} \left\lfloor \sqrt{k} \right\rfloor$$

5. Solve for x:

$$x\lfloor x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor\rfloor = 122.$$