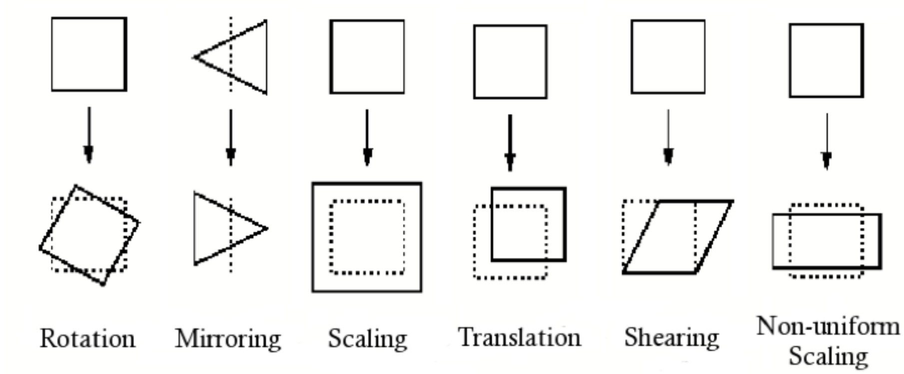
Affine Transformations

ORMC

05/05/24

1 What's an Affine Transformation?

Definition 1.1. An affine transformation is a function $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(x) = Ax + b$, where A is an invertible matrix (see last week) and b is a vector.



Problem 1.2. Show that $f : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation if and only if f is bijective and for all s, t with $s + t = 1$ and all $x, y \in \mathbb{R}^n$,

$$f(sx + ty) = sf(x) + tf(y).$$

Definition 1.3. We say that two sets in $\mathbb{R}^n$ are *affine-equivalent* if there is an affine transformation that sends one to the other.

Problem 1.4. Show that if T is a triangle in $\mathbb{R}^2$, then it is affine-equivalent to an equilateral triangle.

Problem 1.5. The *medians* of a triangle are the three line segments connecting a corner to a midpoint of the opposite side.

a) Show that the three medians of the triangle intersect at the same point.

b) Let C' divide the side AB in the ratio $1 : 7$ and A' divide the side CB in the same ratio. Prove that the intersection point of AC' and CA' belongs to the median from the vertex B.

Hint: if you remember barycentric coordinates, feel free to solve the problem using them.

Problem 1.6. Show that a quadrilateral is affine-equivalent to a square if and only if it is a parallelogram.

Problem 1.7. An ellipse is a plane curve surrounding two focal points, such that for all points on the curve, the sum of the two distances to the focal points is a constant. Show that every ellipse is affine-equivalent to the unit circle.

Hint: use a movement of the plane to put the focal points at $(a, 0)$ *and* $(-a, 0)$ *and write the equation of the ellipse.*

Problem 1.8. Suppose you are given an ellipse and its focal points. Use a ruler and a compass to find any of the triangles largest-by-area inscribed in it.

Hint: Transform the ellipse into the unit circle.

Problem 1.9. Show that every hyperbola is affine-equivalent to $x^2 - y^2 = 1$, and show that that hyperbola is affine-equivalent to $xy = 1$.

2 Problems from *Problem Solving Through Problems* and *Putnam and Beyond*

Some of these problems are best solved by using affine transformations, some can be solved just using vector addition.

Problem 2.1. Prove that the midpoints of the sides of a quadrilateral form a parallelogram.

Problem 2.2. The sides of AD, AB, CB, CD of the quadrilateral $ABCD$ are divided by the points E, F, G, H so that $AE : ED = AF : FB = CG : GB = CH : HD$. Prove that $EFGH$ is a parallelogram.

Problem 2.3. On the sides of an arbitrary parallelogram $ABCD$, squares are constructed lying exterior to it. Prove that their centers M_1, M_2, M_3, M_4 are themselves the vertices of a square.

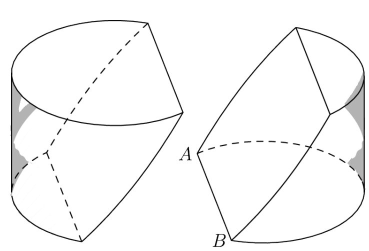
Hint: look for equal triangles.

Problem 2.4. With the chord PQ of a hyperbola as diagonal, construct a parallelogram whose sides are parallel to the asymptotes. Prove that the other diagonal of the parallelogram passes through the center of the hyperbola.

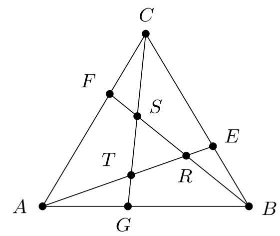
Hint: Show that the hyperbola is affine-equivalent to the "hyperbola" $xy = 1$.

3 Competition Problems

Problem 3.1 (AIME 2015 Problem 15)**.** A block of wood has the shape of a right circular cylinder with radius 6 and height 8, and its entire surface has been painted blue. Points A and B are chosen on the edge of one of the circular faces of the cylinder so that $\overset{\frown}{AB}$ on that face measures 120°. The block is then sliced in half along the plane that passes through point A, point B, and the center of the cylinder, revealing a flat, unpainted face on each half. The area of one of these unpainted faces is $a \cdot \pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a + b + c$.



Problem 3.2 (Putnam 2001 A4)**.** Triangle ABC has an area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.



Hint: What happens to the area of a triangle after you do an affine transformation?

Problem 3.3 (Putnam 1994 A2)**.** Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.