ORMC AMC 10/12 Group
Week 6: Algebra

May 5, 2024

1 Warm-Ups

1. **(2022 AMC 8 #3)** When three positive integers a, b, and c are multiplied together, their product is 100. Suppose $a < b < c$. In how many ways can the numbers be chosen?

2. **(2022 AMC 8 #14)** In how many ways can the letters in **BEEKEEPER** be rearranged so that two or more **E**s do not appear together?

3. **(2022 AMC 8 #25)** A cricket randomly hops between 4 leaves, on each turn hopping to one of the other 3 leaves with equal probability. After 4 hops, what is the probability that the cricket has returned to the leaf where it started?

4. **(2006 AMC 10A #11)** Which of the following describes the graph of the equation $(x+y)^2 = x^2+y^2$?

   (A) the empty set **(B)** one point **(C)** two lines **(D)** a circle **(E)** the entire plane

5. **(2006 AMC 12A #15)** Suppose $\cos x = 0$ and $\cos(x + z) = 1/2$. What is the smallest possible positive value of z?

   (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{5\pi}{6}$ (E) $\frac{7\pi}{6}$

2 Facts/Theorems

In order to solve all the exercises on this worksheet, you should be comfortable with:

- Setting up and solving systems of equations/inequalities
- Binomial theorem, esp. $(a+b)^2 = a^2 + 2ab + b^2$
- Simon's favorite factoring trick
- Graphic equations and functions
- Finding prime factorizations
- Arithmetic sequences & series
- AM-GM inequality
- Working with complex numbers
- $(distance) = (rate) \times (time)$
- Vieta's formulas

Solve directly for the expression the problem asks for; do not solve for each variable in the problem individually. Problems are often written in a way that makes them easier to solve this way, and since it involves fewer steps, it is less error prone.

2.1 Proof of AM-GM

One thing that has been mentioned on previous worksheets, but hasn't been formally shown to you, is the fact that AM-GM holds not only for 2 variables, but for any number.

To recap, the 2-variable case is that 2 nonnegative real numbers a, b always satisfy:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

And in general, if we have n real numbers $a_1, \dots, a_n$, they satisfy:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

The standard strategy for proving this is very interesting. We first show, inductively, that if the formula holds for $n = 2^k$, then it also holds for $n = 2^{k+1}$. This covers all powers of 2, but leaves gaps between the powers of 2. So, we fill in those gaps by showing that if the formula holds for n, then it also holds for $n-1$.

2.1.1 $(2^k \implies 2^{k+1})$

Inductive hypothesis: the formula holds when $n = 2^k$. Then, suppose we have nonnegative reals $a_1, \dots, a_n, \dots, a_{2n}$. By the inductive hypothesis,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}, \qquad \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n} \geq \sqrt[n]{a_{n+1} a_{n+2} \cdots a_{2n}}$$

Applying the 2-variable case:

$$\frac{\frac{a_1+a_2+\cdots+a_n}{n} + \frac{a_{n+1}+a_{n+2}+\cdots+a_{2n}}{n}}{2} \geq \sqrt{\frac{a_1 + a_2 + \cdots + a_n}{n} \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n}} \geq \sqrt[2n]{a_1 a_2 \cdots a_{2n}}$$

2.2 $(n \implies n-1)$

This involves a clever trick to see that if the formula holds for n, then $n-1$ is just a special case. Given positive reals $a_1, \dots, a_{n-1}$, define $a_n = \sqrt[n-1]{a_1 \cdots a_{n-1}}$. Then,

$$\frac{a_1 + \cdots + a_{n-1}}{n} + \frac{\sqrt[n-1]{a_1 \cdots a_{n-1}}}{n} = \frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} = \left((a_1 \cdots a_{n-1})^{1+1/(n-1)} \right)^{1/n}$$
$$\implies \frac{a_1 + \cdots + a_{n-1}}{n} \geq \frac{n-1}{n} \sqrt[n-1]{a_1 \cdots a_{n-1}} \implies \frac{a_1 + \cdots + a_{n-1}}{n-1} \geq \sqrt[n-1]{a_1 \cdots a_{n-1}}.$$

This completes the proof.

3 Exercises

1. **(2014 AMC 10A #11)** A customer who intends to purchase an appliance has three coupons, only one of which may be used:

   Coupon 1: 10% off the listed price if the listed price is at least \$50

   Coupon 2: \$20 off the listed price if the listed price is at least \$100

   Coupon 3: 18% off the amount by which the listed price exceeds \$100

   For which of the following listed prices will coupon 1 offer a greater price reduction than either coupon 2 or coupon 3?

2. **(2006 AMC 10B #12)** The lines $x = \frac{1}{4}y + a$ and $y = \frac{1}{4}x + b$ intersect at the point $(1, 2)$. What is $a + b$?

3. **(2006 AMC 10B #13)** Joe and JoAnn each bought 12 ounces of coffee in a 16 ounce cup. Joe drank 2 ounces of his coffee and then added 2 ounces of cream. JoAnn added 2 ounces of cream, stirred the coffee well, and then drank 2 ounces. What is the resulting ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee?

4. **(2003 AMC 10A #13)** The sum of three numbers is 20. The first is four times the sum of the other two. The second is seven times the third. What is the product of all three?

5. **(2023 AMC 10B #14)** How many ordered pairs of integers (m, n) satisfy the equation $m^2+mn+n^2 = m^2n^2$?

6. **(2017 AMC 10A #14)** Every week Roger pays for a movie ticket and a soda out of his allowance. Last week, Roger's allowance was A dollars. The cost of his movie ticket was 20% of the difference between A and the cost of his soda, while the cost of his soda was 5% of the difference between A and the cost of his movie ticket. To the nearest whole percent, what fraction of A did Roger pay for his movie ticket and soda?

   (A) 9% **(B)** 19% **(C)** 22% **(D)** 23% **(E)** 25%

7. **(2015 AMC 10A #15)** Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10%?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

8. **(2014 AMC 10A #15)** David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

9. **(2002 AMC 10A #16)** Let $a+1=b+2=c+3=d+4=a+b+c+d+5$. What is $a+b+c+d$?

10. **(2015 AMC 10A #16)** If $y+4=(x-2)^2, x+4=(y-2)^2$, and $x \neq y$, what is the value of x^2+y^2?

11. **(2007 AMC 10A #17)** Suppose that m and n are positive integers such that $75m = n^3$. What is the minimum possible value of $m+n$?

    (A) 15 (B) 30 (C) 50 (D) 60 (E) 5700

12. **(2002 AMC 10B #19)** Suppose that $\{a_n\}$ is an arithmetic sequence with

$$a_1 + a_2 + \cdots + a_{100} = 100 \text{ and } a_{101} + a_{102} + \cdots + a_{200} = 200.$$

What is the value of $a_2 - a_1$?

(A) 0.0001 (B) 0.001 (C) 0.01 (D) 0.1 (E) 1

13. **(2015 AMC 10A #20)** A rectangle with positive integer side lengths in cm has area A cm^2 and perimeter P cm. Which of the following numbers cannot equal $A + P$?

(A) 100 **(B)** 102 **(C)** 104 **(D)** 106 **(E)** 108

14. **(2002 AMC 10A #22)** A set of tiles numbered 1 through 100 is modified repeatedly by the following operation: remove all tiles numbered with a perfect square, and renumber the remaining tiles consecutively starting with 1. How many times must the operation be performed to reduce the number of tiles in the set to one?

15. **(2006 AMC 12A #10)** For how many real values of x is $\sqrt{120 - \sqrt{x}}$ an integer?

(A) 3 **(B)** 6 **(C)** 9 **(D)** 10 **(E)** 11

16. **(2007 AMC 12A #11)** A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with the terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime factor that always divides S?

(A) 3 (B) 7 (C) 13 (D) 37 (E) 43

17. **(2004 AMC 12A #11)** The average value of all the pennies, nickels, dimes, and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?

18. **(2000 AMC 12 #11)** Two non-zero real numbers, a and b, satisfy $ab = a - b$. Which of the following is a possible value of $\frac{a}{b} + \frac{b}{a} - ab$?

$$\textbf{(A)}\ -2 \qquad \textbf{(B)}\ \frac{-1}{2} \qquad \textbf{(C)}\ \frac{1}{3} \qquad \textbf{(D)}\ \frac{1}{2} \qquad \textbf{(E)}\ 2$$

19. **(2003 AMC 12A #12)** Sally has five red cards numbered 1 through 5 and four blue cards numbered 3 through 6. She stacks the cards so that the colors alternate and so that the number on each red card divides evenly into the number on each neighboring blue card. What is the sum of the numbers on the middle three cards?

20. **(2000 AMC 12 #12)** Let A, M, and C be nonnegative integers such that $A + M + C = 12$. What is the maximum value of $AMC + AM + MC + AC$?

21. **(2004 AMC 12B #13)** If $f(x) = ax + b$ and $f^{-1}(x) = bx + a$ with a and b real, what is the value of $a + b$?

22. **(2002 AMC 12B #13)** The sum of 18 consecutive positive integers is a perfect square. The smallest possible value of this sum is

(A) 169 (B) 225 (C) 289 (D) 361 (E) 441

23. **(2013 AMC 12A #16)** A, B, C are three piles of rocks. The mean weight of the rocks in A is 40 pounds, the mean weight of the rocks in B is 50 pounds, the mean weight of the rocks in the combined piles A and B is 43 pounds, and the mean weight of the rocks in the combined piles A and C is 44 pounds. What is the greatest possible integer value for the mean in pounds of the rocks in the combined piles B and C?

(A) 55 **(B)** 56 **(C)** 57 **(D)** 58 **(E)** 59

24. **(2004 AMC 12B #16)** A function f is defined by $f(z) = i\overline{z}$, where $i = \sqrt{-1}$ and $\overline{z}$ is the complex conjugate of z. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

(A) 0 (B) 1 (C) 2 (D) 4 (E) 8

25. **(2004 AMC 12B #17)** For some real numbers a and b, the equation

$$8x^3 + 4ax^2 + 2bx + a = 0$$

has three distinct positive roots. If the sum of the base-2 logarithms of the roots is 5, what is the value of a?

(A) -256 (B) -64 (C) -8 (D) 64 (E) 256