Probability

Useful formulas

- $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$
- The number of ways to choose k objects from n is $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Generating functions and crazy dice

1. In this problem, we will construct two six sided dice X and Y, with positive integers such that X+Y has the same probability distribution as the sum of two regular dice. The generating function $f_X(x)$for a random variable X is a polynomial whose x^n coefficient is the probability P(X=n). For example, if Z is a regular die, $f_Z(x) = \frac{1}{6}(x + x^2 + \ldots + x^6)$.

   (a) Show that if Z_1 and Z_2 are any two independent dice, $f_{Z_1+Z_2}(x) = f_{Z_1}(x)f_{Z_2}(x)$. For example, what happens if Z_1 and Z_2 are regular dice?

   (b) Factor the expression $(\frac{1}{6}(x+x^2+\ldots+x^6))^2$ (You may use the formula $1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n-1}{x-1}$)

   (c) Split the factors in (b) and expand to show that the dice $(1, 2, 2, 3, 3, 4)$ and $(1, 3, 4, 5, 6, 8)$ solve our problem. It turns out that this is the only solution to our problem, but showing this is a little tedious.

Binomial coefficients and lattice paths

2. Show that the number of lattice paths from $(0, 0)$ to (a, b) is $\binom{a+b}{a}$.
3. Use problem 2 to prove the following identities:
   (a) $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$
   (b) $\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$
4. Use the binomial theorem $(1+x)^\alpha = 1+\alpha x+\alpha(\alpha-1)x^2/2!+\ldots$ to show that $\frac{1}{\sqrt{1-4x}} = 1 + \binom{2}{1}x + \binom{4}{2}x^2 + \binom{6}{3}x^3 + \ldots$

Catalan numbers and lattice paths

5. Show that the number of lattice paths from $(0, 0$ to (n, n) that do not go above the line $y = x$ is $C_n = \frac{1}{n+1}\binom{2n}{n}$ (Hint: Show that the number that do cross that line is $\binom{2n}{n-1}$). The Catalan numbers are ubiquitous in combinatorics; Professor Richard Stanley has over 200 combinatorial interpretations of them on his website.

Random walk on a line

6. Suppose a drunk leaves a bar and walks up and down the street at random. We model the process by a random walk. We suppose he starts at 0 on the number line. Every second, he takes a step to the left or right with equal probability.
(a) What is the probability he is back at 0 after $2n$ steps, $n \geq 1$? (see problem 2)
(b) What is the expected number of times he returns to 0? (This is the sum over $n \geq 1$ of the numbers in (a); use problem 4)
(c) What is the probability that he comes back to 0 for the first time after $2n$ steps? (see problem 5)

7. (a) Suppose that there is a cliff at 3 and his home is at -2. What is the probability that he reaches home safely?
(b) Same problem, but now suppose he goes left with probability 2/3 and right with probability 1/3.

8. Try doing problem 6, parts (a) and (b) for random walks on a plane and in space and make sense of the following quote: 'A drunk man will find his way home, but a drunk bird may get lost forever.'-Shizuo Kakutani