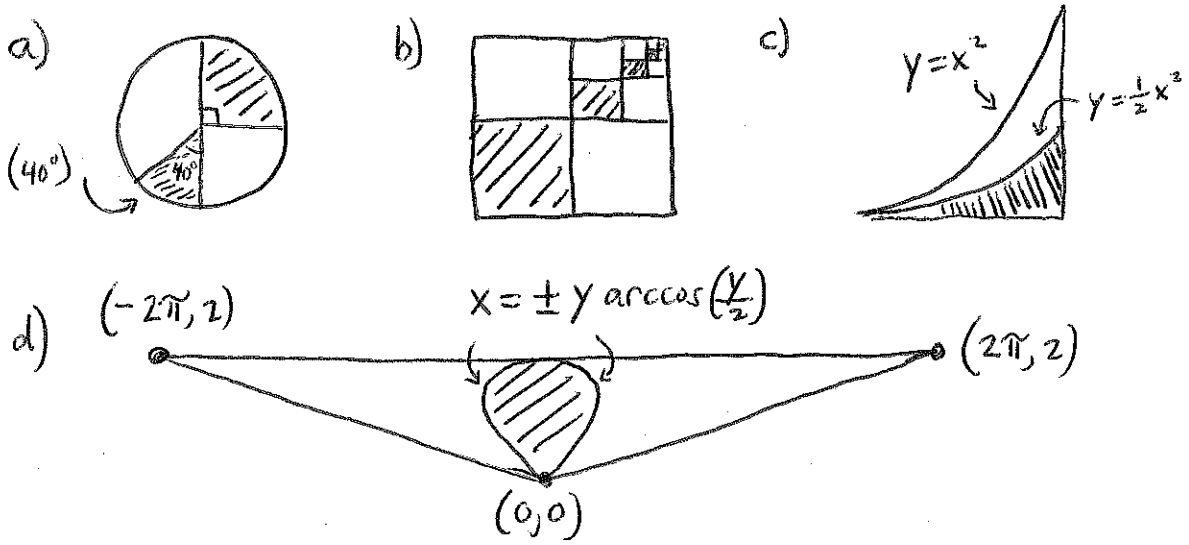
Geometric Probability

What does it mean to choose a "random" point out of a region? When choosing a random point from some geometrical region, if success corresponds to choosing a point from some subset of the region,

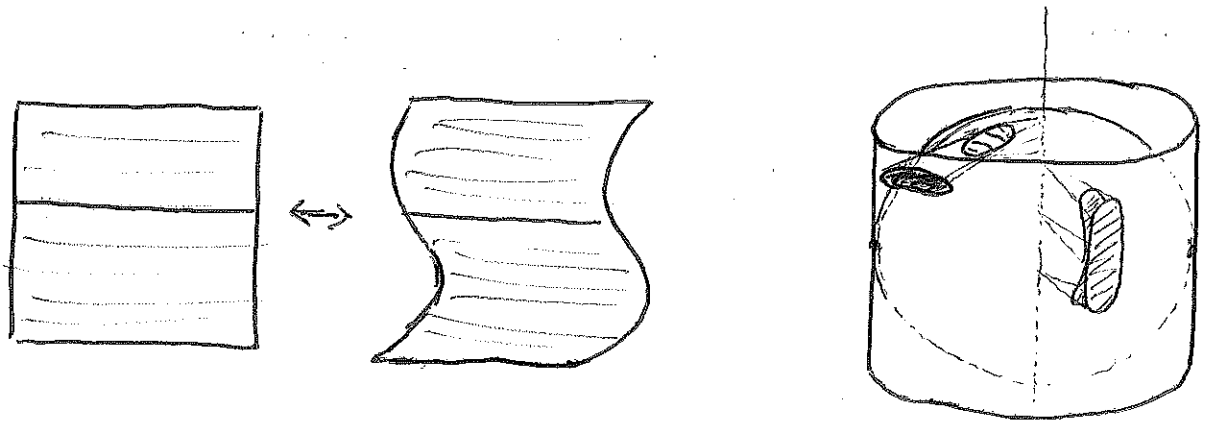
$$\mathbb{P}(\text{success}) = \frac{\text{Size of successful region}}{\text{Size of total region}}$$

1. In each figure, find the probability that a randomly chosen point lies in the shaded region.



Cavalieiri's Principle If two regions have cross-sections of equal size at corresponding levels, then the two regions have the same size.

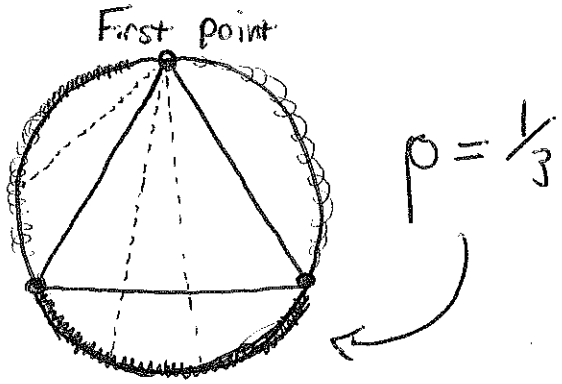
Archimedes' Theorem On the Sphere and Cylinder If a region on a sphere is projected onto a circumscribed cylinder, the projection has the same area.



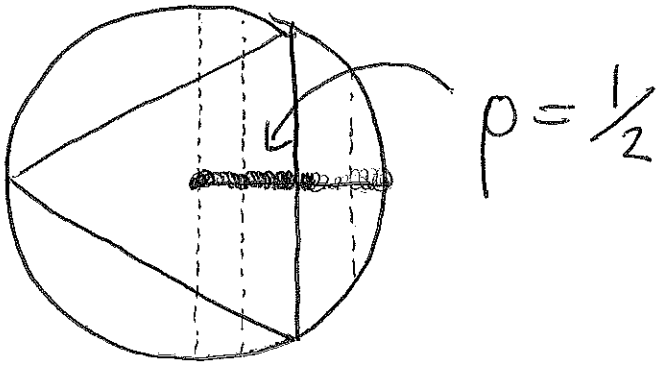
Bertrand's Paradox

If a chord of a circle is chosen randomly, what is the probability that its length is greater than one side of an inscribed equilateral triangle?

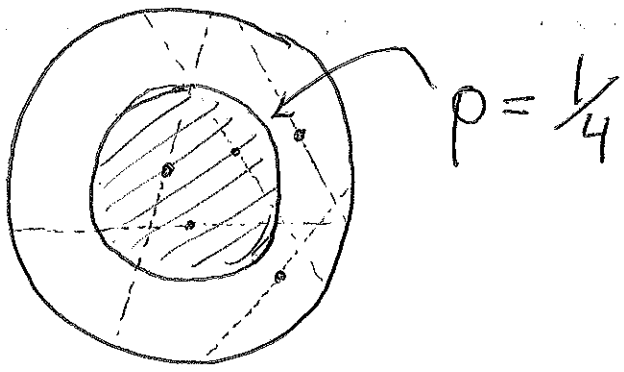
1. Choose a random point on the circle, then a second random point, and connect the two. By symmetry, it doesn't matter where we choose the first point.



2. By symmetry, we can assume without loss of generality the chord is vertical, and lies in the right half of the circle.

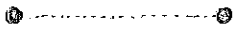


3. Choose a random point in the circle, and construct the chord which has the given point as its midpoint.

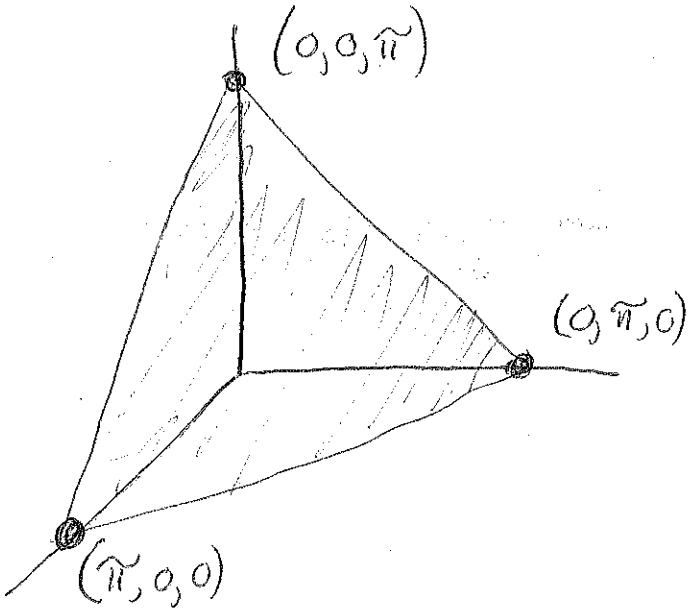


Is a random triangle acute?

1. Two points are indicated below. Shade in the region where a third point can be drawn so as to make an obtuse triangle.



2. The angles of a triangle must sum to π. On the figure below, the set of possible angles is drawn. Sketch the region corresponding to acute triangles. What fraction of the area is it?



3. Three diameters of a circle are chosen randomly, and one endpoint of each is then chosen randomly.

   (a) When is the triangle formed by connecting the three points obtuse?

   (b) What is the probability that this happens?

4. Three points are chosen randomly on a circle. What is the probability that they are the vertices of an acute triangle?

5. A stick is broken at two random places. What is the probability that the three resulting pieces can form a triangle?

6. A random point of an equilateral triangle is chosen, and connected by perpendiculars to each of the three sides. What is the probability that none of the perpendiculars is longer than $1/2$?

7. A dog catches a stick while doing a backflip, snapping it at a random place. The dog picks up the larger piece and does another backflip, but breaks this piece at a random place as well. What is the probability that the three resulting pieces can form a triangle?

8. Three diameters of a circle are chosen, and a triangle is formed with sides parallel to the given diameters. What is the probability that the triangle is acute?

9. (*) Given that numbers $x, y, z \geq 0$ satisfy $x^2 + y^2 + z^2 = 1$, what is the probability that $x + y + z \geq \sqrt{2}$?

   The following problems may require Calculus.

10. A random point is chosen on a circle, and then two random angles between 0 and π are chosen. The chords that make the given angles with a tangent to the circle at the first point determine two more points. What is the probability that the triangle formed by these three points is acute?

11. A random point is chosen on a circle, then a second point a random distance counterclockwise from the first, then a third a random distance counterclockwise from the second so that the third point is not past the first. What is the probability the three chosen points form an acute triangle?