AMC 8 Training: Combinatorics

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1 Warm Up

Let R be a set of nine distinct integers. Six of the elements are 2, 3, 4, 6, 9, and 14. What is the the number of possible values of R?

(A) 4 B) 5 C) 6 D) 7 E) 8

How many positive 4 digit integers have four different digits, wehre the leading digit is not zero, the integer is a multiple of 5, and 5 is the largest digit?

(A) 24 (B) 48 (C) 60 (D) 84 (E) 108

A baseball league consists of two four-team divisions. Each team plays every other team in its division N games. Each team plays every team in the other division M games with $N > 2M$ and $M > 4$. Each team plays a 76-game schedule. How many games does a team play within its own division?

(A) 36 (B) 48 (C) 54 (D) 60 (E) 72

2 Permutations and Combinations

2.1 Permutations

In a set of elements, a permutation is simply a rearrangement of its elements. The number of ways to re-order all n objects is simply $n! = 1 \times 2 \cdots \times n$. To re-order r objects of of the set of n objects when order matters, we can use a permutation:

$$\begin{aligned} {}_nP_r &= n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

This permutation formula often works best when all the objects in the set are different. If you have identical objects, you can switch their places and still have the same permutation. In this case, we need to make sure we do not over count by dividing by $n!$ for each repeated object.
If the i-th element has a_i copies, then the total number of permutations is:

$$\frac{(a_1 + a_2 + \cdots + a_n)!}{a_1!a_2!\ldots a_n!}.$$

Examples:

1. Using only the letters given in each name, find the number of unique permutations of letters you can make.

a. PHOEBE

b. KYLAR

c. CONNOR

d. XAVIER

2. Let S be the set of permutations of the sequence $1, 2, 3, 4, 5$ for which the first term is not 1. A permutation is chosen randomly from S. Find the probability that the second term is 2.

2.2 Combinations

A combination is a way to choose a subset from a set of elements. In a combination, order does not matter. The number of ways to choose r out of n objects where order doesn't matter is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

We can also find this using Pascal's Triangle to iteratively find the *binomial coefficient*.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

3 Other Counting Strategies

3.1 Complementary Counting

In some cases, it's easier to count the elements that aren't needed in a set. It would be easier to count the number of elements and subtract those that you don't need. This is called *complementary counting*.

3.2 Stars and Bars

Stars and Bars is a strategy used to sort identical elements into distinct groups. It utilizes stars ($*$) and bars ($|$).

If we had 7 people who like chocolate ice cream and 3 people who have vanilla ice cream, we can sort them as such: $*******|***$

The divider bar $|$ represents the separation between the two groups, while the stars $*$ represent the elements in each group.

In general, the number of ways to split n identical objects split among m people is equivalent to chosing $m-1$ dividers among $n+(m-1)$ positions for a total of
$\binom{n+m-1}{m-1}$.

4 Probability!

Permutations and combinations are incredibly helpful when we look at probability. The probability of an event is always between 0 and 1 inclusive. Probability is the number of ways that a certain outcome can occur divided by the total number of possible outcomes

Probability questions usually come in the following forms:
FAIR VS UNFAIR COINS
For fair coins, the probability to flip a head or a tail is $\frac{1}{2}$
For unfair coins, if the probability to flip a head is p, then the probability to flip a tail is 1 - p.
DICE
A fair six-sided dice has a probability on landing on any specific number.

There are a couple rules we can follow with probability:
Multiplication Principle If event A can occur in m ways and then event B can occur in n ways, no matter what happens in event A, then event A followed by event B can occur in m $\times$ n ways.

Binomial Probability Formula

$$P(r) = \binom{n}{r} p^r q^{n-r},$$

where P is the probability of exactly r successes. p is the probability of success in one trial and q is the probability of failure ($p + q = 1$). n is the number of independent trials. $0 \le r \le n$.

Example: Alex and Bob are throwing darts at a board independently. Each person is throwing four times. The probabilities of hitting the board are $\frac{2}{3}$ and $\frac{3}{4}$ for Alex and Bob, respectively. What is the probability that Alex hits the board exactly two times and Bob hit the board exactly 3 times?

5 Practice

A single bench section at a school event can hold either 7 adults or 11 children. When N bench sections are connected end to end, an equal number of adults and children seated together will occupy all the bench space. What is the least possible positive integer value of N?

(A) 9 (B) 18 (C) 27 (D) 36 (E) 77

A positive integer divisor of 12! is chosen at random. The probability hat the divisor chosen is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 3 (B) 5 (C) 12 (D) 18 (E) 23

For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

(A) 14 (B) 16 (C) 18 (D) 19 (E) 21

A child builds towers using identically shaped cubes of different colors. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

(A) 24 (B) 288 (C) 312 (D) 1260 (E)40320

The numbers $1, 2, \ldots, 9$ are randomly placed into the 9 squares of a 33 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

(A) $\frac{1}{21}$ (B) $\frac{1}{4}$ (C) $\frac{5}{63}$ (D) $\frac{2}{21}$ (E) $\frac{1}{7}$

How many ways can a student schedule 3 mathematics courses – algebra, geometry, and number theory – in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.)

(A) 3 (B) 6 (C) 12 (D) 18 (E) 24

Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

(A) 105 (B)114 (C) 190 (D) 210 (E) 380

How many distinguishable arrangements are there of 1 brown tile, 2 green tiles, and 3 yellow tiles in a row from left to right? (Tiles of the same color are indistinguishable.)

(A) 210 (B) 420 (C) 630 (D) 840 (E) 1050

There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same?

(A) $\frac{29}{128}$ (B) $\frac{23}{128}$ (C) $\frac{1}{4}$ (D) $\frac{35}{128}$ (E) $\frac{1}{2}$

Each vertex of convex pentagon ABCDE is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?

How many four-digit positive integers have at least one digit that is a 2 or a 3?

Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

How many odd positive 3-digit integers are divisible by 3 but do not contain the digit 3?

The probability that Chris will win the first set of a tennis match is $\frac{2}{3}$ and that he will win the second is $\frac{1}{2}$. Assuming independence of the two sets, what is the probability that he wins both sets?

A bag of marbles consists of 8 red and 7 black marbles. Three marbles are chosen at random without replacement. What is the probability that they are the same color? Express your answer as a common fraction.

A point E is chosen at random from within square ABCD. Express as a fraction the probability that ABE is obtuse.

A bug starts at a vertex of an equilateral triangle. On each move, it randomly selects one of the two vertices where it is not currently located, and crawls along a side of the triangle to that vertex. Find the probability that the bug moves to its starting vertex on its tenth move.