Matrices

ORMC

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1 Linear functionals

Problem 1.1. There are n citizens living in Oddtown. Their main occupation was forming various clubs, which at some point started threatening the very survival of the city. In order to limit the number of clubs, the city council decreed the following innocent-looking rules:

- Each club has to have an odd number of members.
- Every two clubs must have an even number of members in common.

Prove that under these rules, it is impossible to form more clubs than n, the number of citizens.

Hint: map clubs to vectors in F_2^n and show they are linearly independent

Problem 1.2 (Moscow Math Olympiad 1995)**.** Several light bulbs are lit on a panel, and there are several buttons. Pressing a button changes the state of the light bulbs connected to it. It is known that for any set of light bulbs, there exists a button connected to an odd number of bulbs in this set. Prove that by pressing the buttons, you can turn off all the light bulbs.

2 Matrices

Definition 2.1. To multiply two matrices, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The entry in the resulting matrix at position (i, j) is obtained by multiplying each element of the i-th row of the first matrix by the corresponding element of the j-th column of the second matrix and then summing the products.

For example, consider multiplying matrix A with matrix B to form matrix C, where $C = AB$. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then C will be an $m \times p$ matrix with:

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Here is a visual representation with the i-th row of A and the j-th column of B highlighted:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \ B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ c_{31} & c_{32} & \cdots & c_{3p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}, \ c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + \cdots + a_{2n}b_{n2}$$

Where the element c_{22} is calculated by taking the dot product of the second row of A with the second column of B.

Problem 2.2. Prove that matrix multiplication is associative. That is, for any three matrices A, B, and C, where the multiplication is defined, prove that:

$$(AB)C = A(BC)$$

Hint: Consider the general element of the matrix $(AB)C$ and show that it is equal to the corresponding element in $A(BC)$ by using the definition of matrix multiplication.

Problem 2.3. Compute

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 9 \\ 10 \\ 11 \\ 12 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \begin{bmatrix} 9 & 10 & 11 & 12 \end{bmatrix}$$

Problem 2.4. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

a) Show that

$$A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix},$$

where F_n is the n-th Fibonacci number.

b) Show that

$$A = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{\sqrt{5}-1}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{\sqrt{5}+1}{2\sqrt{5}} \end{bmatrix}$$

c) Show that

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{\sqrt{5}-1}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{\sqrt{5}+1}{2\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

d) Prove the Binet formula:

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

Problem 2.5 (Fredholm's Alternative.)**.** Consider a system of linear equations with fixed coefficients $a_{i,j}$, $(1 \leq j, i \leq n)$. You can represent it with matrices like this:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Show that one of the following options always holds:

1. for any $b_1, \ldots, b_n$, the system has exactly one solution (in particular, for $b_1 = \ldots = b_n = 0$ only the trivial solution exists);
2. for some $b_1, \ldots, b_n$, the system is inconsistent, and for some (including nontrivial) there are infinitely many solutions.

In the first case, we call the matrix *invertible.*
Hint: Gaussian elimination

Problem 2.6. It is known that

$$\begin{aligned} a_1 - 4a_2 + 3a_3 &\geq 0, \\ a_2 - 4a_3 + 3a_4 &\geq 0, \\ &\vdots \\ a_{99} - 4a_{100} + 3a_1 &\geq 0, \\ a_{100} - 4a_1 + 3a_2 &\geq 0. \end{aligned}$$

Let $a_1 = 1$; what values can numbers $a_2, \ldots, a_{100}$ take?

Problem 2.7. The cells of a 10×10 square are filled with numbers so that the number in each internal cell is equal to the average of its four neighbors.

a) Prove that if the boundary cells contain only 0's, all the cells contain 0's.

b) Prove that the numbers in internal cells are uniquely determined by the numbers in the boundary cells. *Hint: Use Fredholm's alternative.*

Problem 2.8. a) Prove that for any distinct $x_1, x_2, \ldots, x_{n+1}$ and any i from 1 to $n+1$ there exists a polynomial of degree n equal to zero in all x_j's except for x_i.

b) Use the first part to prove the Interpolation theorem: For any $n+1$ bivariate data points $(x_0, y_0), \ldots, (x_n, y_n) \in \mathbb{R}^2$, where no two x_j are the same, there exists a unique polynomial $p(x)$ of degree at most n that interpolates these points, i.e., $p(x_0) = y_0, \ldots, p(x_n) = y_n$.

After solving Problem 2.8, you might enjoy seeing the first time Chinese remainder theorem was introduced: https://yawnoc.github.io/sun-tzu/iii/26

Problem 2.9. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a vector. The Vandermonde matrix V generated by $\mathbf{v}$ is given by:

$$V = \begin{bmatrix} 1 & v_1 & v_1^2 & \cdots & v_1^{n-1} \\ 1 & v_2 & v_2^2 & \cdots & v_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & v_n & v_n^2 & \cdots & v_n^{n-1} \end{bmatrix}$$

a) Prove that V is invertible. *Hint: use Problem 2.8*

b) If you know what determinants are, compute the determinant of this matrix.

3 Putnam problems

Problem 3.1 (1991 Putnam A2)**.** Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

Problem 3.2 (2008 Putnam A2)**.** Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled.

Alan wins if the resulting matrix is invertible; Barbara wins if it is not. Which player has a winning strategy?