"Irrational Numbers Part II": The Number e

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1 Warm-up: Problems with the same answer (ish)

Recall that $\mathbb{P}$ denotes the probability of an event, and $\mathbb{E}$ denotes the expected value of a random variable. All random events are uniformly random (meaning they occur with equal probability).

Problem 1 *(Derangement Problem) N people arrive at a party, and leave their hats at the door, which are put into boxes. But the organizer forgot to label the boxes, so each person is given a random person's hat at the end of the night. For $N = 4$, find*

$$\frac{1}{\mathbb{P}(\textbf{\textit{nobody}}\ \textit{receives their own hat})}$$

If you have additional time and a calculator, try $N = 5, 6$ as well.

Problem 2 *(Bernoulli Trials) Alice and Bob play a game, where Bob will roll an N-sided die N times (for as many N as he likes). Alice wins if* ***any*** *of the die land on 1, and Bob wins otherwise. For $N = 5$, find*

$$\frac{1}{\mathbb{P}(\textit{Bob wins})}$$

If you have additional time and a calculator, try $N = 10, 15, 20$ as well.

Problem 3 *(Variant of Bernoulli Trials) Randomly throw N balls into N boxes. For $N = 4$, find*

$$\frac{N}{\mathbb{E}(\textit{number of empty boxes})}$$

If you have additional time and a calculator, try $N = 5, 6, 7$ as well.

Problem 4 *(Stirling's Approximation) Given a finite set of positive integers $S = \{k_1, k_2, k_3, \ldots, k_n\}$, we define their* ***geometric mean*** *G_S to be*

$$G_S = \sqrt[n]{k_1 k_2 k_3 \ldots k_n}$$

For $N = 10$, find

$$\frac{N}{G_{\{1,2,3,\ldots,N\}}}$$

If you have additional time and a calculator, try $N = 20, 30, 40$ as well.

Problem 5 *(Challenge) We pick random numbers between 0 and 1 as follows. First pick a random real number between 0 and 1, and add it to a list. Then pick another random real number between 0 and 1, and if it's smaller than the last one on the list, we stop picking numbers. If it's bigger than the last one, we add it to the list and keep picking. Find*

$$1 + \mathbb{E}(\textit{number of numbers in the list})$$

Problem 6 *(Challenge) Randomly remove adjacent pairs of integers $(m, m + 1)$ from $\{1, 2, 3, \ldots, N\}$ until there are no more adjacent integers. For large values of N, find*

$$\frac{1}{\sqrt{1 - \frac{1}{N}\mathbb{E}(\textit{number of integers removed})}}$$

2 The definitions of e

A few weeks ago, we encountered a certain definition of Euler's number e. We saw that

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

Its decimal approximation is $e = 2.71828\dots$

Problem 7 *Compare the answers you got in Problems 1-4 to the decimal approximation for* e*. We will show that each of these values is approximately* e *for large* N*, so how good was your approximation?*

Problem 8 *Show that*

$$\frac{1}{e} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \dots$$

(Hint: Multiply the first few terms, and see what happens to "the rest".)

Problem 9 *Show that the answer to Problem* 1 *is approximately* e *for large* N*.*

Problem 10 *(Bonus) Show that*

$$e^k = \frac{k^0}{0!} + \frac{k^1}{1!} + \frac{k^2}{2!} + \frac{k^3}{3!} + \frac{k^4}{4!} + \dots$$

for any positive integer n. (Hint: Induction)

Next up are the answers to Problems 2 and 3, for which we'll need a different equivalent formula for e.

Problem 11 *Show the **binomial theorem**, which states that*

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

(Hint: Induction works, but for a shorter proof, consider the amount of ways to get $a^k b^{n-k}$ in the expansion of $(a+b)^n$.)

Problem 12 *Show that for a fixed k and large n,*

$$\frac{\binom{n}{k}}{n^k} \approx \frac{1}{k!}$$

(Hint: Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. What do you get when you expand the factorials?)

Problem 13 *Show that for large n,*

$$e \approx \left(1 + \frac{1}{n}\right)^n$$

Also, using the result of Problem 10*, show that for large n,*

$$e^k \approx \left(1 + \frac{k}{n}\right)^n$$

Problem 14 *Show that the answers to Problems* 2 *and* 3 *are approximately e for large N.*

3 The natural logarithm and approximations

To show that the answer to Problem 4 is also e, we need better ways of approximating very large numbers, such as the logarithm. Recall that the *base b logarithm* (for a base $b > 1$) is defined by

$$\log_b(b^x) = b^{\log_b(x)} = x$$

for all nonnegative x. (In other words, $\log_b$ is the inverse function of b^x.) We call the base e logarithm the *natural* logarithm, denoted ln.

Problem 15 *Evaluate the following logarithms (continued on the following page).*

- $\log_2(4)$

- $\log_2(1/16)$

- $\log_3(27)$

- $\log_4(8)$

- $\log_{10}(0.001)$

- $\ln(1/e)$

Problem 16 *Show that*

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

Use this fact to prove that

$$\log_b(x^n) = n \log(b)$$

Problem 17 *Using the previous problem, rewrite* $\ln(n!)$ *as a sum.*

The sum from the previous problem is a well-known expression, which approximately equals

$$\ln(1) + \ln(2) + \cdots + \ln(n) \approx n \ln n - n + 1$$

Stirling used a better approximation to derive his famous formula for $n!$:

Theorem 1 *(Stirling's Formula) For large n,*

$$\ln(n!) \approx n \ln n - n + \frac{1}{2} \ln(2\pi n)$$

or equivalently (more famously),

$$n! \approx \sqrt{2\pi n} \frac{n^n}{e^n}$$

The proof of both of these formulas involves calculus (which you should read on your own once you take that class). But we can get a similar approximation just fine without it.

Problem 18 *Show that*

$$n \ln n - n < \ln(n!) < n \ln n$$

(Hint: For the right inequality, what is $n \ln n$ the natural log of? For the left inequality, find an $n!$ in the expression for e^n and rearrange.)

Problem 19 *Using Stirling's formula (Theorem 1), show that the answer to Problem 4 is approximately e for large N.*