Vector spaces

ORMC

04/21/24

1 Vector spaces

Definition 1.1. Vector space over a field F is a set V with two operations called vector addition and scalar multiplication.

- Vector addition assigns to any two vectors v and w in V a third vector in V which is commonly written as $v + w$, and called the sum of these two vectors.
- Scalar multiplication, assigns to any scalar a in F and any vector v in V another vector in V, which is denoted av.

These operations should satisfy some axioms:

Axiom	**Meaning**
Associativity of vector addition	$u + (v + w) = (u + v) + w$
Commutativity of vector addition	$u + v = v + u$
Identity element of vector addition	There exists an element $0 \in V$, called the zero vector, such that $v + 0 = v$ for all $v \in V$.
Inverse elements of vector addition	For every $v \in V$, there exists an element $-v \in V$, called the additive inverse of v, such that $v + (-v) = 0$.
Compatibility of scalar multiplication with field multiplication	$a(bv) = (ab)v$
Identity element of scalar multiplication	$1v = v$, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(u + v) = au + av$
Distributivity of scalar multiplication with respect to field addition	$(a + b)v = av + bv$

Examples of vector spaces are $\mathbb{R}^2$ and $\mathbb{R}^3$ – our familiar plane and space using $\mathbb{R}$ as a base field. A less trivial example is the space of all quadratic polynomials (possibly with the zero coefficient in x^2). The last example has a base field F_2. It is the space of all functions from a set $\{1, 2, 3, 4, 5, 6, 7\}$ to F_2. Vector addition is a pointwise addition and scalar multiplication is a pointwise multiplication.

Definition 1.2. Given a set G of elements of a F-vector space V, a linear combination of elements of G is an element of V of the form

$$a_1\mathbf{g}_1 + a_2\mathbf{g}_2 + \cdots + a_k\mathbf{g}_k,$$

where $a_1, \ldots, a_k \in F$ and $\mathbf{g}_1, \ldots, \mathbf{g}_k \in G$. The scalars $a_1, \ldots, a_k$ are called the coefficients of the linear combination.

Definition 1.3. The elements of a subset G of a F-vector space V are said to be linearly independent if no element of G can be written as a linear combination of the other elements of G. Equivalently, they are linearly independent if two linear combinations of elements of G define the same element of V if and only if they have the same coefficients. Also equivalently, they are linearly independent if a linear combination results in the zero vector if and only if all its coefficients are zero.

Definition 1.4. Basis (pl. bases) is a maximal by inclusion linearly independent set in a vector space. For example, vectors $(1,,0,0)$ and $(0,1,0)$ and $(0,0,1)$ form a basis in $\mathbb{R}^2$ and $(1,2)$ and $(2,1)$ form a basis of F_5^2 (but not of F_3).

Lemma 1.5. *All bases of a vector space have the same size. This size is called a dimension of V.*

Problem 1.6. Consider all linear combinations of vectors

$$\begin{bmatrix}1\\0\\0\\0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\1\\1\end{bmatrix}$$

in F_2^7. Prove that they form a vector space. What is the dimension of this space? Prove that any two elements of it differ in at least 3 coordinates.

Problem 1.7. A 10×10 board is initially colored white. One can simultaneously switch all the colors in a row or column (white cells become black and black cells become white).

a) Prove that one can not use these operations to achieve a position where only one cell is black.

b) Introduce a vector space on all accessible configurations.

c) What is the dimension of this vector space? What is the number of accessible configurations?

Problem 1.8. There are n citizens living in Oddtown. Their main occupation was forming various clubs, which at some point started threatening the very survival of the city. In order to limit the number of clubs, the city council decreed the following innocent-looking rules:

- Each club has to have an odd number of members.
- Every two clubs must have an even number of members in common.

Prove that under these rules, it is impossible to form more clubs than n, the number of citizens.
Hint: map clubs to vectors in F_2^n and show they are linearly independent

We saw this problem on a number theory worksheet, but maybe now's a good time to revisit it:

Problem 1.9. The prime factorizations of $r+1$ positive integers (with $r > 1$) together involve only r primes. Prove that there is a subset of these integers whose product is a perfect square.

Problem 1.10. a) Prove that for any distinct $x_1, x_2, \ldots, x_{n+1}$ and any i from 1 to $n+1$ there exists a polynomial of degree n equal to zero in all x_j's except for x_i.

b) Use the first part to prove the Interpolation theorem: For any $n+1$ bivariate data points $(x_0, y_0), \ldots, (x_n, y_n) \in \mathbb{R}^2$, where no two x_j are the same, there exists a unique polynomial $p(x)$ of degree at most n that interpolates these points, i.e., $p(x_0) = y_0, \ldots, p(x_n) = y_n$.

After solving Problem 1.10, you might enjoy seeing the first time Chinese remainder theorem was introduced: https://yawnoc.github.io/sun-tzu/iii/26

Problem 1.11. 24 students solved 25 problems. The teacher has a 24×25 table indicating which students solved which problems. It turns out that each problem was solved by at least one student. Prove that it is possible to mark some of the problems with a "check" so that each student has solved an even number (which could be zero) of the checked problems.

2 Olympiad problems

Problem 2.1 (Moscow Math Olympiad 1995)**.** Several light bulbs are lit on a panel, and there are several buttons. Pressing a button changes the state of the light bulbs connected to it. It is known that for any set of light bulbs, there exists a button connected to an odd number of bulbs in this set. Prove that by pressing the buttons, you can turn off all the light bulbs.

Problem 2.2 (Tournament of Towns 2002/2003)**.** Play the game :

https://www.chiark.greenend.org.uk/∼sgtatham/puzzles/js/flip.html.

Show that for the 4×4 board not any board state is reachable from the empty one.

Problem 2.3 (USAMO 2008 Problem 6)**.** At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2^k for some positive integer k).

Hint: Suppose that you can change from one valid seating arrangement to another by having exactly the mathematicians in the set S switch rooms. Interpret the set of all possible S as a vector space.

Problem 2.4 (Putnam 2003 B1)**.** Do there exist polynomials $a(x), b(x), c(y), d(y)$ with real coefficients such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Hint: The set of polynomials $p(x)$ that can be written as $r \cdot a(x) + s \cdot b(x)$, for $r, s \in \mathbb{R}$, is a vector space over $\mathbb{R}$. What is its dimension? Is this enough for all polynomials $p(x)$ you can get by plugging some value of y into $1 + x + x^2$?

Problem 2.5 (Putnam 2009 B4)**.** Say that a polynomial with real coefficients in two variables, x, y, is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over $\mathbb{R}$. Find the dimension of V.

Hint: For some $k \leq 2009$, can you find the dimension of the set of balanced polynomials where every term has degree exactly k?