OLGA RADKO MATH CIRCLE, SPRING 2024: ADVANCED 3

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Worksheet 4:

Let $\mathbb{F}$ be a field of characteristic different to 2 or 3.

Let $x^3 + ax + b$ be a cubic polynomial with coefficients in $\mathbb{F}$ that has no repeated roots. An *elliptic curve* over $\mathbb{F}$ is defined as the set of points (x, y) in $\mathbb{F}^2$ satisfying the equation

$$y^2 = x^3 + ax + b,$$

together with a single point denoted O and called the point at infinity.

Problem 4.1:

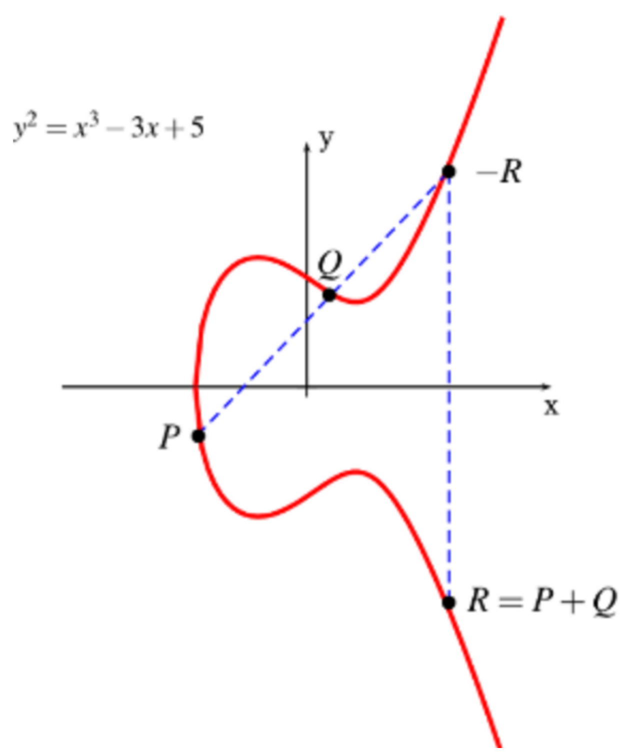
(1) How many points does the elliptic curve given by the equation $y^2 = x^3 - x$ over $\mathbb{F}_5$ have?
(2) How many points does the elliptic curve given by the equation $y^2 = x^3 + x$ over $\mathbb{F}_5$ have?
(3) Show that no elliptic curve over $\mathbb{F}_5$ can have more than 11 points.

Solution 4.1:

For the following problem our field will be the real numbers, thus we can draw the curves.
Let E be an elliptic curve, let P and Q be two points in E. We will define $P+Q$ and $-P$ by the following rules.

(1) If $P = O$, then $-P := O$ and $P+Q := Q$, so in the following cases we will assume that no point is the point at infinity.
(2) If the point P has coordinates (x, y), then the point $-P$ is given by the the coordinates $(x, -y)$
(3) If P and Q have different x-coordinates, then the line $l = \overline{PQ}$ intersects E at a third point R (in case l is tangent to E at P or Q, we define R to be the point of tangency). We define $P+Q = -R$.
(4) If $Q = -P$, then $P+Q := O$.
(5) If $P = Q$, then let l be the tangent line to E at P, let R be the third point of intersection of l and E. We define $P+Q := -R$.

These definitions also work in the case of any field, if one takes care of what a tangent curve means in those cases. An example of this can be seen in the following picture:



Problem 4.2:

For the following problem our field will be the real numbers, thus we can draw the curves.

Let L be some line that is not paralel to the y-axis, show the following equalities:

(1) Assume $L \cap E$ are three different points A, B, C. Then
$$A + B + C = O$$
(2) Assume $L \cap E$ consists of two points A, B, where L is tangent to E at A. Then $A + A + B = O$.
(3) Assumme $L \cap E = \{A\}$. Then $A + A + A = O$.

Solution 4.2:

The objective of the next problem is to find a formula for the point $P+Q$, in terms of a, b and the coordinates of P and Q.

Let E be the elliptic curve over $\mathbb{R}$ given by the equation $y^2 = x^3 + ax + b$, and $P = (x_1, y_1)$, $Q = (x_2, y_2)$. Assume that $x_1 \neq x_2$.

Let $y = \alpha x + \beta$ be the equation of the line l passing through P and Q.

Let $R = (x_3, y_3)$ be the third point of intersection of l and E.

Problem 4.3:

(1) Show that the x-coordinate of the intersection points of l and E satisfy the equation:
$$x^3 - (\alpha x + \beta)^2 + ax + b = 0$$
(2) Show that $x_3 = \alpha^2 - x_1 - x_2 = (\frac{y_2 - y_1}{x_2 - x_1})^2 - x_1 - x_2$.
(3) Compute the coordinates of $P+Q$.

This can be taken as the definition of $P+Q$, when $P \neq Q$, in any field.

Solution 4.3:

Let E be the elliptic curve over $\mathbb{R}$ given by the equation $y^2 = x^3 + ax + b$, and $P = (x_1, y_1)$. Let $y = \alpha x + \beta$ be the equation of the line l tangent to E passing through the point P.

The tangent line to E at P has slope $\alpha = \frac{3x_1^2+a}{2y_1}$.

Problem 4.4:

(1) Show that the x coordinates of the intersection points of l and E satisfy the equation:

$$x^3 - (\alpha x + \beta)^2 + ax + b = 0$$

(2) Show that $x_3 = \alpha^2 - 2x_1 = (\frac{3x_1^2+a}{2y_1})^2 - 2x_1$.

(3) Compute the coordinates of $P + P$.

This formula can be taken as the definition of $P + P$ for an elliptic curve over any field.

Solution 4.4:

One can think of the elliptic curve given by equation $y^2 = x^3 + ax + b$ with the point at infinity O, as a projective curve in $\mathbb{P}^2_{\mathbb{F}}$, with equation

$y^2z = x^3 + axz^2 + bz^3$

Problem 4.5

Show that there exists a bijection between the points of the elliptic curve E (O and points (x, y) in $y^2 = x^3+ax+b$) and points in the projective plane satisfying the equation $y^2z = x^3 + axz^2 + bz^3$.

What point in the projective plane corresponds to O?

Solution 4.5

Problem 4.6:

Show that $P + Q + R = O$ if and only if P, Q, R are the intersection points of a line and E, when we think of it as a projective curve.

Which cases are we missing if we only consider lines in affine space?

Solution 4.6:

Let $L_1, \dots, L_6$ be homogeneous linear polynomials in three variables,i.e. $L_i = 0$ is a projective line in $\mathbb{P}^2$. Let $X = L_1L_2L_3$ and $Y = L_4L_5L_6$. Assume that $\{X = 0\} \cap \{Y = 0\}$ are nine different points.

For the following problem, you may use that if a elliptic curve (seen as a projective curve) passes through 8 of the points in $\{X = 0\} \cap \{Y = 0\}$ then the intersection of the degree 3 curve and $\{X = 0\} \cap \{Y = 0\}$ is exactly the 9 points of $\{X = 0\} \cap \{Y = 0\}$.

Problem 4.7:

Assuming that the points $P, Q, -(P + Q), O, R, -R, -P, -(Q + R), (P + Q) + R$ are all different, show that $(P + Q) + R = P + (Q + R)$.

Hint: Find 6 lines $L_1, \dots, L_6$, $X = L_1L_2L_3$ and $Y = L_4L_5L_6$ such that $X \cap Y = \{P, Q, -(P+Q), O, R, -R, -P, -(Q+R), (P + Q) + R\}$.

Solution 4.7:

Problem 4.8:

Show that the points of an elliptic curve with the addition defined before form an abelian group i.e. show that the operation satisfies associativity, commutativity, there exists a neutral element and any element has an inverse.

Solution 4.8:

An element P of a group $(E, +_E)$ is said to have order d if d is the smallest positive integer such that

$$dP = \underbrace{P +_E \cdots +_E P}_{d \text{ times}} = 0$$

Problem 4.9:

Find the order of the point $(2, 3)$ on the elliptic curve $y^2 = x^3 + 1$.

Solution 4.9:

Problem 4.10:

Let P be a point in an elliptic curve E, different from the point at infinity.

(1) Show that a point has order 2 if and only if it is on the x-axis
(2) Show that a point P has order 3 if and only if the tangent to E at P does not contain any other point of E.
(3) Can you find a geometric description of the points of order 4 in E?

Solution 4.10:

Problem 4.11:

Let p be an odd prime number, different from 3. Show that there are $p+1$ different points in the following elliptic curves over $\mathbb{F}_p$

(1) $y^2 = x^3 - x$, for $p \equiv 3 \pmod 4$.
(2) $y^2 = x^3 - 1$, for $p \equiv 2 \pmod 3$.

Solution 4.11:

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