"Irrational Numbers Part II": The Number e

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1 Warm-up: Problems with the same answer (ish)

Recall that $\mathbb{P}$ denotes the probability of an event, and $\mathbb{E}$ denotes the expected value of a random variable. All random events are uniformly random (meaning they occur with equal probability).

Problem 1 *(Derangement Problem) N people arrive at a party, and leave their hats at the door, which are put into boxes. But the organizer forgot to label the boxes, so each person is given a random person's hat at the end of the night. For $N = 4$, find*

$$\frac{1}{\mathbb{P}(\boldsymbol{nobody} \text{ receives their own hat})}$$

If you have additional time and a calculator, try $N = 5, 6$ as well.

Problem 2 *(Bernoulli Trials) Alice and Bob play a game, where Bob will roll an N-sided die N times (for as many N as he likes). Alice wins if **any** of the die land on 1, and Bob wins otherwise. For $N = 5$, find*

$$\frac{1}{\mathbb{P}(Bob \ wins)}$$

If you have additional time and a calculator, try $N = 10, 15, 20$ as well.

Problem 3 *(Variant of Bernoulli Trials) Randomly throw N balls into N boxes. For $N = 4$, find*

$$\frac{N}{\mathbb{E}(\textit{number of empty boxes})}$$

If you have additional time and a calculator, try $N = 5, 6, 7$ as well.

Problem 4 *(Stirling's Approximation) Given a finite set of positive integers $S = \{k_1, k_2, k_3, \ldots, k_n\}$, we define their **geometric mean** G_S to be*

$$G_S = \sqrt[n]{k_1 k_2 k_3 \ldots k_n}$$

For $N = 10$, find

$$\frac{N}{G_{\{1,2,3,\ldots,N\}}}$$

If you have additional time and a calculator, try $N = 20, 30, 40$ as well.

Problem 5 *(Challenge) We pick random numbers between 0 and 1 as follows. First pick a random real number between 0 and 1, and add it to a list. Then pick another random real number between 0 and 1, and if it's smaller than the last one on the list, we stop picking numbers. If it's bigger than the last one, we add it to the list and keep picking. Find*

$$1 + \mathbb{E}(\textit{number of numbers in the list})$$

Problem 6 *(Challenge) Randomly remove adjacent pairs of integers $(m, m+1)$ from $\{1, 2, 3, \ldots, N\}$ until there are no more adjacent integers. For large values of N, find*

$$\frac{1}{\sqrt{1 - \frac{1}{N}\mathbb{E}(\textit{number of integers removed})}}$$

2 The definitions of e

A few weeks ago, we encountered a certain definition of Euler's number e. We saw that

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

Its decimal approximation is $e = 2.71828\ldots$

Problem 7 *Compare the answers you got in Problems 1-4 to the decimal approximation for e. We will show that each of these values is approximately e for large N, so how good was your approximation?*

Problem 8 *Show that*

$$\frac{1}{e} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \ldots$$

(Hint: Multiply the first few terms, and see what happens to "the rest".)

Solution: There are many ways to check that the first few terms multiply to approximately 1. For the rest of the terms, we have the estimates

$$\left| \pm \frac{1}{n!} + \pm \frac{1}{(n+1)!} + \pm \frac{1}{(n+2!)} \right| \leq \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{1}{(n+2!)} < \frac{1}{(n-1)!}$$

so that the error of these estimates are at most $e/(n-1)!$, and hence approach zero.

Problem 9 *Show that the answer to Problem 1 is approximately e for large N.*

Solution: By the Principal of Inclusion-Exclusion, we can count the number of derangements of a set of N objects. First note that if a permutation is not a derangement, then it sends at least one thing to itself. There are $\binom{N}{k}k!$ permutations that send k things to themselves, so there are

$$N! - \left(\frac{N!}{1!} - \frac{N!}{2!} + \frac{N!}{3!} - \ldots + (-1)^{N-1} \frac{N!}{N!} \right)$$

derangements of a set of N objects, and dividing by the total $N!$ permutations gives a probability approximately $1/e$.

Problem 10 *(Bonus) Show that*

$$e^k = \frac{k^0}{0!} + \frac{k^1}{1!} + \frac{k^2}{2!} + \frac{k^3}{3!} + \frac{k^4}{4!} + \cdots$$

for any positive integer n. (Hint: Induction)

Solution: The base case $k = 1$ is by definition. Now assume the formula holds for k. Since $e^{k+1} = e \cdot e^k$, we consider multiplying the two series. We can group the terms with denominators $i!(n-i)!$, so that we get

$$\frac{k^n}{n!} + \frac{k^{n-1}}{1!(n-1)!} + \cdots + \frac{k}{(n-1)!1!} + \frac{1}{n!} = \frac{k^n + \binom{n}{1}k^{n-1} + \cdots + \binom{n}{n-1}k + 1}{n!} = \frac{(k+1)^n}{n!}$$

(see the Binomial Theorem below). Since we can write this for each n, this completes the induction.

Next up are the answers to Problems 2 and 3, for which we'll need a different equivalent formula for e.

Problem 11 *Show the **binomial theorem**, which states that*

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

(Hint: Induction works, but for a shorter proof, consider the amount of ways to get $a^k b^{n-k}$ in the expansion of $(a+b)^n$.)

Solution: Each $a^k b^{n-k}$ receives an a from k of the factors and a b from the other factors. This is equivalent to choosing k of the factors to receive an a from, for which there are $\binom{n}{k}$ ways.

Problem 12 *Show that for a fixed k and large n,*

$$\frac{\binom{n}{k}}{n^k} \approx \frac{1}{k!}$$

(Hint: Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. What do you get when you expand the factorials?)

Solution: We write

$$\frac{\binom{n}{k}}{n^k} = \frac{n!}{n^k k!(n-k)!} = \frac{n(n-1)\ldots(n-k+1)}{n^k k!}$$

$$= \frac{1}{k!}\frac{n}{n}\frac{n-1}{n}\ldots\frac{n-k+1}{n} = \frac{1}{k!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right) \approx 0$$

since $1/n \approx 0$ for large n.

Problem 13 *Show that for large n,*

$$e \approx \left(1 + \frac{1}{n}\right)^n$$

Also, using the result of Problem 10, show that for large n,

$$e^k \approx \left(1 + \frac{k}{n}\right)^n$$

Solution: By the Binomial Theorem,

$$\left(1 + \frac{1}{n}\right)^n = 1 + \frac{\binom{n}{1}}{n} + \frac{\binom{n}{2}}{n^2} + \cdots + + \frac{\binom{n}{n}}{n^n} \approx \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \approx e$$

for large n, by the previous problem. The proof for e^k is similar.

Problem 14 *Show that the answers to Problems 2 and 3 are approximately e for large N.*

Solution: Since trials are independent, the probability in Problem 2 is $(1 - \frac{1}{N})^N \approx 1/e$. Also, by linearity of expectation, the expected value in Problem 3 is $N(1 - \frac{1}{N})^N \approx N/e$.

3   The natural logarithm and approximations

To show that the answer to Problem 4 is also e, we need better ways of approximating very large numbers, such as the logarithm. Recall that the *base b logarithm* (for a base $b > 1$) is defined by

$$\log_b(b^x) = b^{\log_b(x)} = x$$

for all nonnegative x. (In other words, $\log_b$ is the inverse function of b^x.) We call the base e logarithm the *natural* logarithm, denoted ln.

Problem 15 *Evaluate the following logarithms (continued on the following page).*

- $\log_2(4)$

    Solution: 2

- $\log_2(1/16)$

  Solution: -4

- $\log_3(27)$

  Solution: 3

- $\log_4(8)$

  Solution: 1.5

- $\log_{10}(0.001)$

  Solution: -3

- $\ln(1/e)$

  Solution: -1

Problem 16 *Show that*

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

Use this fact to prove that

$$\log_b(x^n) = n \log(b)$$

Solution: Write $x = b^{e_1}, y = b^{e_2}$. Then $b^{e_1 + e_2} = b^{e_1} b^{e_2} = xy$, so $\log_b(xy) = \log_b(b^{e_1 + e_2}) = e_1 + e_2 = \log_b(x) + \log_b(y)$. The second part follows by repeatedly applying the first.

Problem 17 *Using the previous problem, rewrite* $\ln(n!)$ *as a sum.*

Solution: $\ln(n!) = \ln(1) + \ln(2) + \ln(3) + \cdots + \ln(n)$

The sum from the previous problem is a well-known expression, which approximately equals

$$\ln(1) + \ln(2) + \cdots + \ln(n) \approx n \ln n - n + 1$$

Stirling used a better approximation to derive his famous formula for $n!$:

Theorem 1 *(Stirling's Formula) For large n,*

$$\ln(n!) \approx n \ln n - n + \frac{1}{2} \ln(2\pi n)$$

or equivalently (more famously),

$$n! \approx \sqrt{2\pi n} \frac{n^n}{e^n}$$

The proof of both of these formulas involves calculus (which you should read on your own once you take that class). But we can get a similar approximation just fine without it.

Problem 18 *Show that*

$$n \ln n - n < \ln(n!) < n \ln n$$

(Hint: For the right inequality, what is $n \ln n$ the natural log of? For the left inequality, find an $n!$ in the expression for e^n and rearrange.)

Solution: We have that

$$n^n = n \cdot n \cdot n \ldots n > n \cdot (n-1) \cdot (n-2) \ldots 1 = n!$$

Taking the natural log of both sides gives the right inequality

$$\ln(n!) < \ln(n^n) = n \ln n$$

For the left inequality, we note that

$$\frac{n^n}{n!} < \frac{n^0}{0!} + \frac{n^1}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!} + \cdots = e^n$$

so that

$$n! > \frac{n^n}{e^n}$$

and taking the natural log of both sides gives the left inequality

$$n \ln n - n = \ln\left(\frac{n^n}{e^n}\right) < \ln(n!)$$

Problem 19 *Using Stirling's formula (Theorem 1), show that the answer to Problem 4 is approximately e for large N.*

Solution: The geometric mean of the first N positive integers is $\sqrt[N]{n!}$. By Stirling's Formula,

$$\sqrt[N]{n!} \approx \sqrt[N]{\sqrt{2\pi N} \frac{N^N}{e^N}} = \sqrt[2N]{2\pi N} \frac{N}{e} \approx \frac{N}{e}$$

since $\sqrt[N]{N} \approx 1$ for large N.