Tropical Polynomials

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1 Tropical Arithmetic

In tropical arithmetic, we define new addition and multiplication operations on the real numbers. The **tropical sum** of two numbers is their minimum:

$$x \oplus y = \min(x, y)$$

while the **tropical product** of two numbers is their sum:

$$x \odot y = x + y.$$

1. Which of the following properties hold in tropical arithmetic?

- **Addition is commutative:** $x \oplus y = y \oplus x$.

- **Addition is associative:** $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

- **An additive identity exists:** There exists a real number n such that $x \oplus n = x$ for all real numbers x.

2. Let's expand our number set to include a tropical additive identity. What would be an appropriate name for this new "number"? Give appropriate definitions for the tropical sum and tropical product of this new number with a general real number x and with itself.

3. Which of the following properties hold in tropical arithmetic?

- **Additive inverses exist:** For each number x, there exists a number y such that $x \oplus y = n$, where n is the additive identity.

- **Multiplication is associative:** $(x \odot y) \odot z = x \odot (y \odot z)$.

- **Multiplication is commutative:** $x \odot y = y \odot x$.

- **There exists a multiplicative identity:** There exists a number i such that $x \odot i = x$ for all numbers x.

- **Multiplicative inverses exist:** For each number x not equal to the additive identity, there exists a number y such that $x \odot y = i$, where i is the multiplicative identity.

- **Multiplication distributes over addition:** $x \odot (y \oplus z) = x \odot y \oplus x \odot z$.

4. Complete the tropical addition and multiplication tables below.

$\oplus$	1	2	3	4	∞
1					
2					
3					
4					
∞					

$\odot$	0	1	2	3	4
0					
1					
2					
3					
4					

5. Expand and simplify $f(x) = (x \oplus 2)(x \oplus 3)$, where juxtaposition represents tropical multiplication. Then use your expansion to compute $f(1)$ and $f(4)$.

2 Tropical Polynomials

A **polynomial** is an expression formed by adding and/or multiplying together numbers and copies of a variable x. Every polynomial can be written in the form

$$a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

for some nonnegative integer n and **coefficients** $a_n, \ldots, a_2, a_1, a_0$.

It follows from the **Fundamental Theorem of Algebra** that any non-constant polynomial with real coefficients can be written as a product of polynomials of degree 1 or 2 with **real coefficients**. For example,

$$x^5 + 8x^4 + 17x^3 - 2x^2 - 64x - 160 = (x^2 + 2x + 5)(x - 2)(x + 4)^2.$$

Over the complex numbers, any such polynomial can be factored completely into polynomials of degree 1 with **complex coefficients**. For the example above,

$$x^5 + 8x^4 + 17x^3 - 2x^2 - 64x - 160 = (x + 1 - 2i)(x + 1 + 2i)(x - 2)(x + 4)^2.$$

The factors can be determined by computing the **roots** (or the "zeros") of the polynomial. The polynomial above has roots

$$-1 + 2i, -1 - 2i, 2, -4, -4.$$

We say that the root -4 has **multiplicity** 2.

There is a quadratic formula for determining the roots of a polynomial of degree 2, along with cubic and quartic formulas for degrees 3 and 4. However, starting with degree 5, there is no longer a nice formula which enables us to find the roots of every polynomial. For polynomials of large degree, we generally must settle for approximate roots, found by a computer.

A **tropical polynomial** is an expression formed by (tropically) adding and/or multiplying tropical numbers (i.e., real numbers or ∞) and copies of a variable x. Every tropical polynomial can be written in the form

$$(a_n \odot x^n) \oplus \cdots \oplus (a_2 \odot x^2) \oplus (a_1 \odot x) \oplus (a_0)$$

for some nonnegative integer n and **coefficients** $a_n, \ldots, a_2, a_1, a_0$. (Note that the exponents here represent repeated *tropical* multiplication.) For convenience, we represent tropical multiplication by juxtaposition, in the usual manner:

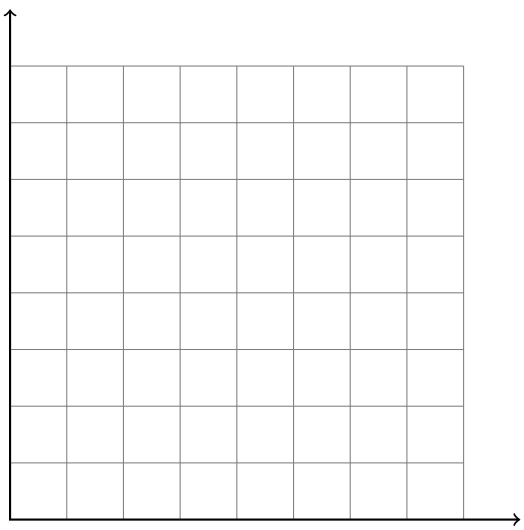
$$a_n x^n \oplus \cdots \oplus a_2 x^2 \oplus a_1 x \oplus a_0.$$

Questions:

- *Can tropical polynomials always be factored completely into polynomials of degree 1?*
- *Is there a tropical quadratic formula for finding the roots of quadratic polynomials? How about a cubic formula?*
- *For polynomials of large degree, must we rely on a computer to find roots, or can we do it by hand?*

2.1 Tropical quadratic polynomials

6. Draw a precise graph of the tropical polynomial $f(x) = x^2 \oplus 1x \oplus 4$. You may find it helpful to first rewrite the tropical polynomial (as an expression involving standard operations) using the definitions of $\oplus$ and $\odot$.



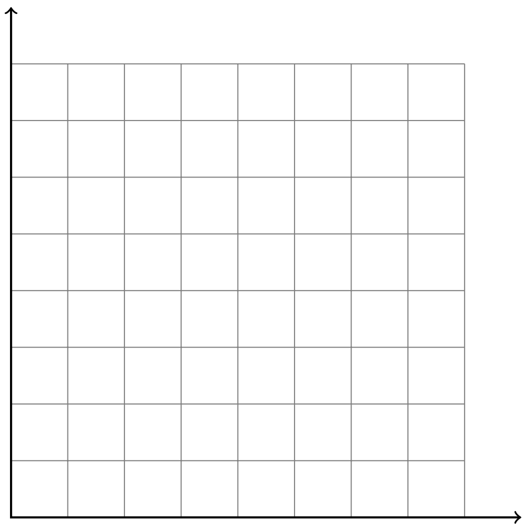
Now, try to factor the tropical polynomial $x^2 \oplus 1x \oplus 4$ into linear (degree 1) factors. In other words, find numbers r and s such that

$$x^2 \oplus 1x \oplus 4 = (x \oplus r)(x \oplus s).$$

These numbers r and s are called the **roots** of the tropical polynomial. (Note that we use $x \oplus r$ and $x \oplus s$ because we do not have a tropical subtraction.)

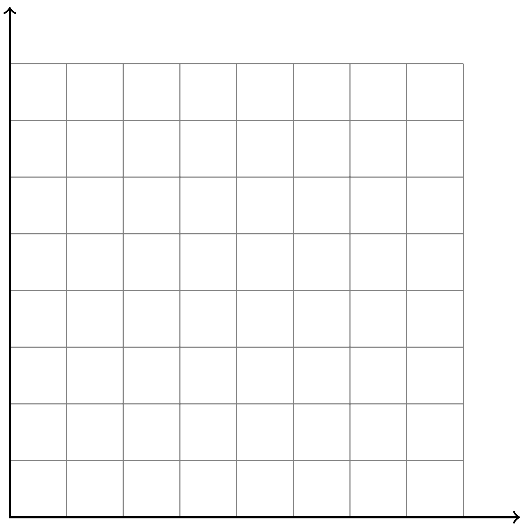
Do you notice any relationship between the graph and the factorization? Can you see the roots in the graph?

7. Graph $f(x) = -2x^2 \oplus x \oplus 8$, and then find a factorization of $f(x)$ in the form $a(x \oplus r)(x \oplus s)$. Can you see the roots r and s in the graph? How are the roots related to the coefficients of $f(x)$?

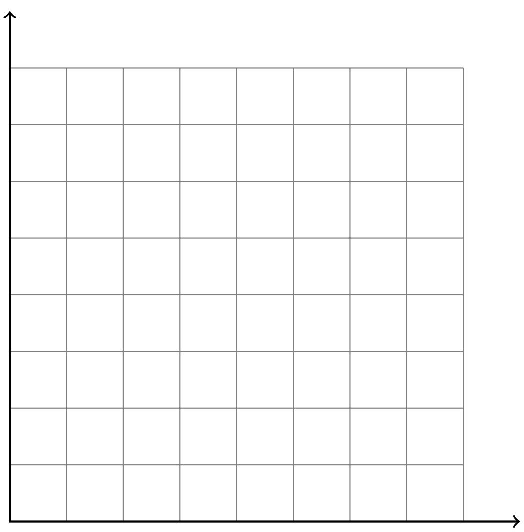


8. Find a tropical polynomial $f(x)$ with a value of 7 for all sufficiently large x and with roots 4 and 5.

9. Graph $f(x) = 1x^2 \oplus 3x \oplus 5$, and then find a factorization in the form $f(x) = a(x \oplus r)(x \oplus s)$. How is this graph different from the previous ones? How is this factorization different from the others? How are the roots related to the coefficients of $f(x)$?



10. Graph $f(x) = 2x^2 \oplus 4x \oplus 4$. Find a factorization in the form $f(x) = a(x \oplus r)(x \oplus s)$, or show that one does not exist.



11. Can you find a tropical polynomial which has the same graph as $f(x) = 2x^2 \oplus 4x \oplus 4$, but which can be factored?

The ***Tropical Fundamental Theorem of Algebra*** *says that, for every tropical polynomial $f(x)$, there is a unique tropical polynomial $\bar{f}(x)$ with the same graph (and therefore determining the same function) which can be factored into linear factors. We sometimes say "the roots of $f(x)$" when we really mean "the roots of $\bar{f}(x)$."*

12. If $f(x) = ax^2 \oplus bx \oplus c$, then $\bar{f}(x) = ax^2 \oplus Bx \oplus c$ for some B. Find a formula for B in terms of a, b, and c. There are two different cases to consider.

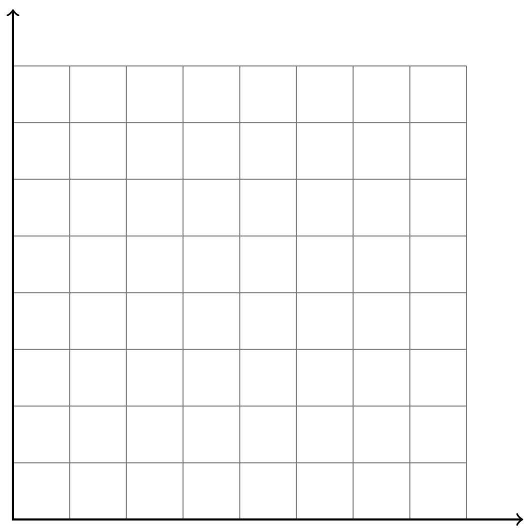
13. State a tropical quadratic formula in terms of a, b, c for the roots x of a tropical polynomial $f(x) = ax^2 \oplus bx \oplus c$ (that is, the roots of the corresponding $\bar{f}$). There are once again two separate cases.

2.2 Tropical cubic polynomials

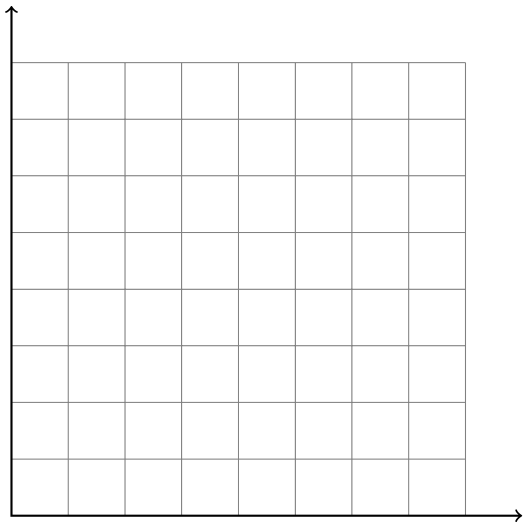
14. For each cubic polynomial below,

- sketch the graph of the polynomial,
- use the graph to find the roots of the polynomial, and
- write (and expand) a product of linear factors with the same graph as the given polynomial.

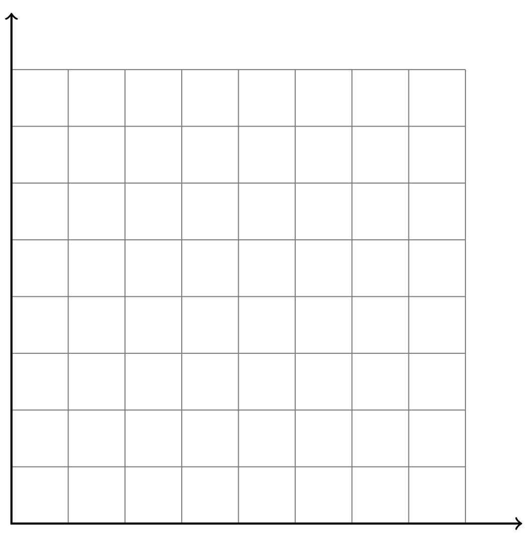
a) $f(x) = x^3 \oplus 1x^2 \oplus 3x \oplus 6$



b) $g(x) = x^3 \oplus 1x^2 \oplus 6x \oplus 6$



c) $h(x) = x^3 \oplus 6x^2 \oplus 6x \oplus 6$



15. If $f(x) = ax^3 \oplus bx^2 \oplus cx \oplus d$, then $\bar{f}(x) = ax^3 \oplus Bx^2 \oplus Cx \oplus d$ for some B and C. With the preceding examples as a guide, find formulas for B and C in terms of a, b, c, and d.

2.3 General tropical polynomials

16. Can you guess the roots of the following polynomial?

$$f(x) = 3x^6 \oplus 4x^5 \oplus 2x^4 \oplus x^3 \oplus 1x^2 \oplus 4x \oplus 5$$

17. If

$$f(x) = a_n x^n \oplus a_{n-1} x^{n-1} \oplus \cdots \oplus a_2 x^2 \oplus a_1 x \oplus a_0,$$

then

$$\bar{f}(x) = a_n x^n \oplus A_{n-1} x^{n-1} \oplus \cdots \oplus A_2 x^2 \oplus A_1 x \oplus a_0.$$

Can you find a formula for each A_j in terms of the a_i? How about formulas for the roots $r_1, r_2, \ldots, r_n$? Can you find a geometric interpretation of these formulas in terms of the points $(-i, a_i)$, for $0 \leq i \leq n$?