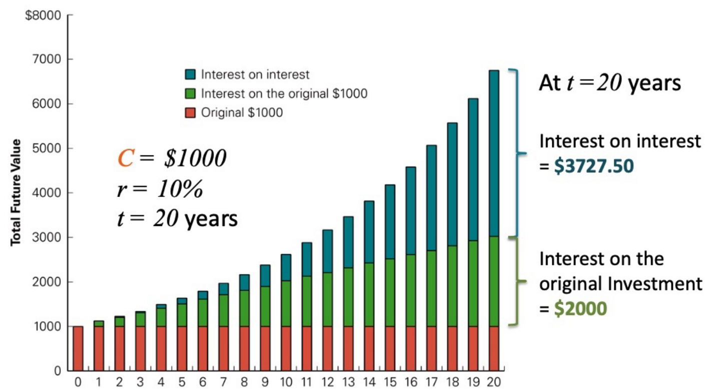
Financial Literacy Part 2: Present Value, Discounting and Multiple Payment Securities

1 Future Value and Compounding: Recap

Future Value is the amount that an investment will be worth at a future date if it is invested at a compounded interest rate.



Future Value for a Single Period is:

$$FV = C(1 + r) \tag{1}$$

Where, FV is future value, C is initial investment, and (1+r) is interest factor)

Future Value for a Multiple Periods are:

$$FV = C(1 + r)^t \tag{2}$$

Where, FV is future value, C is initial investment, and (1+r) is interest factor)

Finding the time it takes to reach a certain amount:

Last week, we covered problems that calculate the future value of an investment given an initial amount (principal) and interest rate. Alternatively, we can find the time it takes to reach a certain future value:

Problem 1.1. *Joe deposits \$1000 to a bank account offering compound interest of % 4 annually. He wants to sit back and enjoy his investment reaching \$ 2000. Use the future value formula to find the interest factor. Recall that interest factor equals* $(1+r)^t$.

Problem 1.2. *Knowing that the interest factor of Joe's investment* $r = 4\%$, *find the time* t *needed for the investment to reach \$2000.*

Problem 1.3. *Deduce the general formula for the time it takes an initial investment,* C, *to reach the target future value,* FV, *at a given interest rate* r. *Hint: Apply some algebraic manipulation to the Future Value formula.*

Problem 1.4. *Maria invests \$800 today in a fund that offers a compound interest rate of 5% per year. Determine the time* t *in years required for Maria's investment to grow to \$3,000. Additionally, calculate how much more money would be in the account after* $t+5$ *years.*

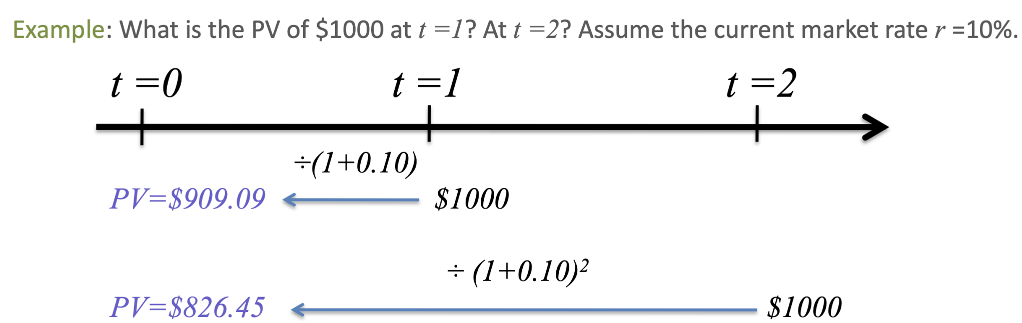
Problem 1.5. *John invests \$2,000 at a compound interest rate of 3.5% per year. Determine the time* t *in years required for John's total investment to exceed \$5,000. Also, analyze the impact if the interest rate increases to 4.5% per year, holding all other factors constant.*

2 Present Value and Discounting

Being the exact opposite of future value, **present value** corresponds to the current value of one or more future cash payments, considering **the opportunity cost** of having the payment in future.

Recall how we discussed the opportunity cost in the past: Opportunity cost is the potential forgone profit from a missed opportunity. In an environment where we can get positive interest by investing our money, a dollar today is worth more than a dollar tomorrow, and we would prefer getting the same amount of money earlier compared to later, as this would allow us to invest the money elsewhere at this moment of time.

Hence, we consider the opportunity cost of getting the payment in the future by using an interest rate, which helps us "discount" the future payment to its present value. This can be better demonstrated through the illustration below:



Do you notice any similarity with the future value calculation? Just as we move forward in time with future value calculations, we do the opposite here, moving backwards in time by dividing at an interest rate, instead of multiplying.

Let's formulate the concept of present value. We already know the formula for future value, in which C is initial investment (principal). Could this resemble Present Value in any way? Let's find out through a problem.

Problem 2.1. *Joe wants to grow his investment to $3000 in 3 years, at an invesment opportunity that gives %20 annual compound interest. Calculate the current amount (or initial, you may as well say) of his investment. Comment on your finding. What does this value signify?*

Problem 2.2. *Going from what you found out in the last problem, deduce the formula for the present value. (Hint: Algebraically manipulate what you have for future value formula.)*

Hece, we find out that Future Value and Present Value are the opposite of each other, where interest factor is multiplied in one of them and divided in the other:

$$FV = PV * (1 + r)^t \tag{3}$$

$$PV = PV/(1 + r)^t \tag{4}$$

Therefore, we can formally define **present value** as future cash payments discounted at an appropriate interest rate.

Problem 2.3. *Bill will receive $5000 in 4 years. Using an annual compound interest rate of 6%, find today's present value of all the payments Bill will receive.*

Problem 2.4. *What is the present value of 1000 due in 10 years if the annual compound interest rate is 5%?*

Interest rate as a macroeconomic concept:

So far, we talked about the interest rate as a value given to us by an investment opportunity, oftentimes, our interest account at a bank account. However, in real life, the interest rate is not an arbitrary number. It is determined by central banks of countries based on multiple factors, most importantly inflation, government policy, and expectations of future economic growth. (We don't need to diverge too much here with macroeconomic principles, but it's good practice to know the current and historical interest put by the FED.)

Historically average interest rates of FED (Federal Reserve, the central bank of the US) for the last 5 years (at the end of each year) have been as follows:

Year	Interest Rate(%)
2019	1.5-1.75
2020	0-0.25
2021	0-0.25
2022	4.25-4.5
2023	5.25-5.5

Table 1: Two Investments

As you can see, interest rates in the real world change quite rapidly due to a multitude of reasons. Hence, we can introduce some problems adjusting to changing interest rates.

Problem 2.5. *What is the present value of 1000 due in 5 years if the annual compound interest rate is 2% for the first year, 1% for the next 2 years, and 5% for the final 3 years*

Problem 2.6. *The parents of three children aged 1, 3, and 6 wish to set up an investment fund that will pay 20,000 to each child upon attainment of age 18. If the investment fund will earn annual effective interest at 10%, what amount must the parents now invest in the trust fund?*

Problem 2.7. *The parents of three children aged 1, 3, and 6 wish to set up an investment fund that will pay 20,000 to each child upon attainment of age 18. If the investment fund will earn annual effective interest at 10%, what amount must the parents now invest in the trust fund?*

Problem 2.8. *Smith has debts of 1500 due now. He proposes to repay them with a single payment of 2000 one year from now. What is the implied annual effective interest rate if the replacement payment is accepted as equivalent to the original debts?*

3 Multiple Payment Securities:

So far, we have covered future value and present value calculations where there is a single payment, whether in terms of an initial investment or a future payment. Oftentimes, we observe situations where multiple payments are made, whether that's us providing monthly contributions to our investment portfolio (as we discussed at the end of last week) or getting monthly payments from an investment tool. When we talk about the future value

of an investment completed over multiple periods (e.g. investing $2000 not in a single payment but over 4 years with $500 yearly investments), we find future value by using the formula:

$$FV = \sum_{t=0}^{n} C * (1+r)^{n-t} \tag{5}$$

where, C is the cash flow of a given period, r is the interest(or discount) rate, t is the year that ranges from 0 to n, and n is the last year in which a cash flow occurs. We don't need to delve too much into this formula, but it's good practice to understand how the sigma summation enables us to sum investments through multiple periods. Likewise, we define the present value

of a cash flow stream as:

$$PV = \sum_{t=0}^{n} \frac{C}{(1+r)^{n-t}} \tag{6}$$

where, C is the cash flow of a given period, r is the interest(or discount) rate, t is the year that ranges from 0 to n, and n is the last year in which a cash flow occurs. Again, we don't need to delve too much into this formula, but it's good practice to understand how the sigma summation enables us to discount multiple future payments into a present value.

Having understood the concept, we will delve into two categories of such payments:

Annuities: Any investment opportunity that pays a fixed sum each year for a specified number of years. We calculate the present value of an annuity through a very practical formula:

$$PV = \frac{C}{r} * (1 - \frac{1}{(1+r)^n}) \tag{7}$$

Problem 3.1. *You are the lucky winner of the \$30 million state lottery. You can take your prize money either as (a) 30 payments of \$1 million per year, or (b) \$15 million paid today. If the interest rate is 8%, which option should you take?*

Problem 3.2. *Mike owes his friend Jane \$5000 as of now. Mike does not have such an amount right now, therefore Mike proposes Jane pay his debt over 5 years with payments of \$1100 each year, totaling \$5500. Suppose the interest rate is 3%. What is the present value of Mike's future payments? Is Mike profiting off this deal or not?*

Perpetuities: Perpetuities are streams of equal cash flows that occur at regular intervals and last forever. Present value of perpetuities are:

$$PV = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r} \tag{8}$$

Problem 3.3. *Suppose you are moving out of town and trying to appoint someone to take care of the garden of your family house. Maintenance of your garden costs \$10,000 every year, forever. The interest rate is 5% per year. How much money should you leave to the gardener of your family house?*

Problem 3.4. *Suppose you want to be involved in charity affairs and create a donation fund that pays out \$10,000 as a scholarship every year to support a student's education. Assuming an interest rate of 4% per year, how much should you donate from the beginning?*

4 Chapter Problems

Problem 4.1. *Using the fact that the geometric mean of a collection of positive numbers is less than or equal to the arithmetic mean, show that if annual compound interest rates over an n-year period are i_1 in the first year, i_2 in the second year, $\ldots$, i_n in the n^{th} year, then the average annual compound rate of interest for the n-year period is less than or equal to $\frac{1}{n} \cdot \sum_{k=1}^{n} i_k$.*

Problem 4.2. *Joe deposits 10 today and another 30 in five years into a fund paying simple interest of 11% per year. Tina will make the same two deposits, but the 10 will be deposited n years from today and the 30 will be deposited $2n$ years from today. Tina's deposits earn an annual effective rate of 10%. At the end of 10 years, the accumulated amount in Tina's deposits equals the accumulated amount of Joe's deposits. Calculate n.*

Problem 4.3. *A magazine offers a one-year subscription for \$15 with renewal the following year at \$16.50. Also offered is a two-year subscription for \$28. What is the annual effective interest rate that makes the two-year subscription equivalent to two successive one-year subscriptions?*

Problem 4.4. *Payments of 200 due July 1, 2020, and 300 due July 1, 2022, have the same value on July 1, 2017, as a payment of 100 made on July 1, 2017, along with a payment made on July 1, 2021. Find the payment needed on July 1, 2021, assuming annual effective interest at a rate of 4%.*

Problem 4.5. *Ed buys a TV from Al for 480 by paying 50 in cash, 100 every three months for one year (four payments of 100), and a final payment in 15 months (three months after the final quarterly payment). Find the amount of the final payment if Al earns a 3-month compound interest rate of 3%. What is the final payment if Al earns a one-month rate of 1%?*

Problem 4.6. *David can receive one of the following two payment streams:*

1. *100 at time 0, 200 at time* n*, and 300 at time* $2n$
2. *600 at time 10*

At an annual effective interest rate of i*, the present values of the two streams are equal. Given* $v^n = 0.75941$*, determine* i*.*

Problem 4.7. *A manufacturer can automate a certain process by replacing 20 employees with a machine. The employees each earn 24,000 per year, with payments on the last day of each month, with no salary increases schedules for the next 4 years. If the machine has a lifetime of 4 years and interest is at a monthly rate of .75%, what is the most the manufacturer would pay for the machine (on the first day of a month) in each of the following cases?*

- *The machine has no scrap value at the end of the 4 years.*
- *The machine has a scrap value of 200,000 after 4 years.*
- *The machine has a scrap value of 15% of its purchase price at the end of the 4 years.*

Problem 4.8. *A contract calls for payments of 750 every 4 months for several years. Each payment is to be replaced by two payments of 367.85 each, one to be made 2 months before, and one to be made at the time of, the original payment. Find the 2-month rate of interest implied by this proposal if the new payment scheme is financially equivalent to the old one.*

Problem 4.9. *Fisheries officials are stocking a barren lake with pike, whose number will increase annually at the rate of 40%. The plan is to prohibit fishing for two years on the lake, and then allow the removal of 5000 pike in each of the third and fourth years so that the number remaining after the fourth year is the same as the original number stocked in the lake. Find the original number, assuming that the stocking takes place at the start of the year and removal takes place at midyear.*

Problem 4.10. *Smith lends Jones 1000 on January 1, 2020, on the condition that Jones repay 100 on January 1, 2021, 100 on January 1, 2022, and 1000 on January 1, 2023. On July 1, 2021, Smith sells to Brown the rights to the remaining payments for 1000, so Jones makes all future payments to Brown. Let j be the 6-month rate earned on Smith's net transaction, and let k be the 6-month rate earned on Brown's net transaction. Are j and k equal? If not, which is larger?*

Problem 4.11. *Smith has debts of 1000 due now and 1092 due two years from now. He proposes to repay them with a single payment of 2000 one year from now. What is the implied annual effective interest rate if the replacement payment is accepted as equivalent to the original debts?*