ORMC AMC 10/12 Group
Week 4: Combinatorics

April 21, 2024

1 Warm-Ups

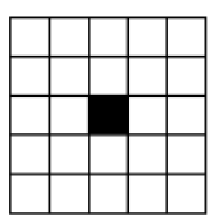
1. **(2004 AMC 10A #10)** Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same?

2. **(2007 AMC 10A #11)** The numbers from 1 to 8 are placed at the vertices of a cube in such a manner that the sum of the four numbers on each face is the same. What is this common sum?

3. **(2006 AMC 10A #13)** A player pays \$5 to play a game. A die is rolled. If the number on the die is odd, the game is lost. If the number on the die is even, the die is rolled again. In this case the player wins if the second number matches the first and loses otherwise. How much should the player win if the game is fair? (In a fair game the probability of winning times the amount won is what the player should pay.)

4. **(2005 AMC 10A #15)** How many positive cubes divide $3! \cdot 5! \cdot 7!$?

5. **(2004 AMC 10A #16)** The 5×5 grid shown contains a collection of squares with sizes from 1×1 to 5×5. How many of these squares contain the black center square?



2 Facts/Theorems

A lot of the exercises will require facts from various topics that we have discussed in previous classes. It would likely be helpful to go through the following list before you start, to make sure you remember all of them:

- Casework
- Burnside's Lemma
- Binomial coefficients
- Principle of Inclusion-Exclusion
- Stars-and-Bars
- Overcounting
- Complementary Counting
- Counting factors of a number
- Modular arithmetic

A couple other useful facts/strategies that have been mentioned, but not used extensively, are:

2.1 Vandermonde's identity

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

Combinatorially, it is easy to see why this is true. Think of the identity as counting the number of ways to pick a size-r committee out of a group of m teachers and n students. We can do this:

1. As the right-hand side suggests: all at once, choosing r people out of the total $m+n$, OR
2. We can break it down into cases where we fix the number of teachers we choose. The left hand side sums over these cases, where we first choose k teachers, and then $r-k$ students.

2.2 Recursion

Recursion is a method of counting which uses the fact that configurations or sequences often depend on previous values. For instance, a problem may ask you to count:

1. How many ways a bug can stay on a certain section of a grid, while taking n steps
2. How many sequences of 0's and 1's of a given length n fit some specific criteria.

Problems like these lend themselves very easily to recursion, since the values/configurations for a specific n can often be written very easily in terms of values/configurations for smaller values of n. This is perhaps best illustrated with an example:

2.2.1 Call a sequence of 0's and 1's "valid" if no 1 ever appears directly next to another 1. How many sequences of length 10 are "valid"?

Notice that if we have a sequence of length 10 which is valid, then its first k elements must also form a valid sequence, for every $k \leq 10$. We use this fact to consider our sequence of length 10 recursively, in terms of its shorter subsequences.

If the final element in the length-10 sequence is a 1, then the one before it must be a 0. Then, all the elements before that can be whatever we want, as long as they form a valid sequence of length 8.

If the final element is 0, then all the elements before it can be whatever we want, as long as they form a valid sequence of length 9.

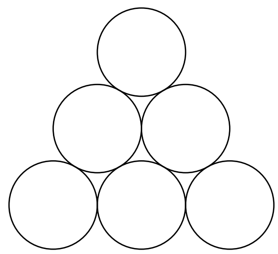
If we denote the number of valid sequences of length n as V_n, this reasoning tells us that $V_n = V_{n-1} + V_{n-2}$. This is the same recursive relation as the Fibonacci numbers F_n. We can easily see that $V_1 = 2$ and $V_2 = 3$, so it is easy to see now that $v_n = F_{n+2}$.

So, the number of valid sequences of length 10 is $V_{10} = F_{12} = \boxed{144}$.

3 Exercises

1. **(2015 AMC 10A #10)** How many rearrangements of *abcd* are there in which no two adjacent letters are also adjacent letters in the alphabet? For example, no such rearrangements could include either *ab* or *ba*.

2. **(2017 AMC 12B #13)** In the figure below, 3 of the 6 disks are to be painted blue, 2 are to be painted red, and 1 is to be painted green. Two paintings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same. How many different paintings are possible?



3. **(2017 AMC 12B #16)** The number $21! = 51,090,942,171,709,440,000$ has over $60,000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

4. **(2005 AMC 10A #18)** Team A and team B play a series. The first team to win three games wins the series. Each team is equally likely to win each game, there are no ties, and the outcomes of the individual games are independent. If team B wins the second game and team A wins the series, what is the probability that team B wins the first game?

5. **(2018 AMC 10A #20)** A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?

6. **(2021 Fall AMC 12B #17)** A bug starts at a vertex of a grid made of equilateral triangles of side length 1. At each step the bug moves in one of the 6 possible directions along the grid lines randomly and independently with equal probability. What is the probability that after 5 moves the bug never will have been more than 1 unit away from the starting position?

7. **(2023 AIME I #1)** Five men and nine women stand equally spaced around a circle in random order. The probability that every man stands diametrically opposite a woman is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

8. **(2015 AMC 10A #22)** Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

9. **(2016 AMC 10B #22)** A set of teams held a round-robin tournament in which every team played every other team exactly once. Every team won 10 games and lost 10 games; there were no ties. How many sets of three teams $\{A, B, C\}$ were there in which A beat B, B beat C, and C beat A?

10. **(2022 AMC 10A #22)** Suppose that 13 cards numbered $1, 2, 3, \ldots, 13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards $1, 2, 3$ are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, $7, 8, 9, 10$ on the fourth pass, and $11, 12, 13$ on the fifth pass. For how many of the 13! possible orderings of the cards will the 13 cards be picked up in exactly two passes?

$$\boxed{7}\ \boxed{11}\ \boxed{8}\ \boxed{6}\ \boxed{4}\ \boxed{5}\ \boxed{9}\ \boxed{12}\ \boxed{1}\ \boxed{13}\ \boxed{10}\ \boxed{2}\ \boxed{3}$$

11. **(2017 AMC 10A #23)** How many triangles with positive area have all their vertices at points (i, j) in the coordinate plane, where i and j are integers between 1 and 5, inclusive?

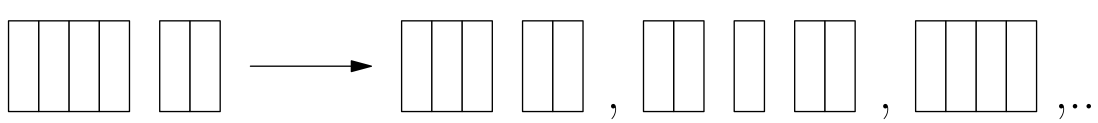
12. **(2022 AMC 10A #24)** How many strings of length 5 formed from the digits 0, 1, 2, 3, 4 are there such that for each $j \in \{1,2,3,4\}$, at least j of the digits are less than j? (For example, 02214 satisfies this condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

13. **(2021 Fall AMC 12B #20)** A cube is constructed from 4 white unit cubes and 4 blue unit cubes. How many different ways are there to construct the $2 \times 2 \times 2$ cube using these smaller cubes? (Two constructions are considered the same if one can be rotated to match the other.)

14. **(2015 AIME I #2)** The nine delegates to the Economic Cooperation Conference include 2 officials from Mexico, 3 officials from Canada, and 4 officials from the United States. During the opening session, three of the delegates fall asleep. Assuming that the three sleepers were determined randomly, the probability that exactly two of the sleepers are from the same country is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

15. **(2018 AMC 12B #22)** Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0,1,2,3,4,5,6,7,8,9\}$. How many such polynomials satisfy $P(-1) = -9$?

16. **(2021 AMC 12B #22)** Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one "wall" among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move: $(3,2), (2,1,2), (4), (4,1), (2,2)$, or $(1,1,2)$.



Arjun plays first, and the player who removes the last brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

(A) $(6,1,1)$ **(B)** $(6,2,1)$ **(C)** $(6,2,2)$ **(D)** $(6,3,1)$ **(E)** $(6,3,2)$

17. **(2017 AMC 10A #25)** How many integers between 100 and 999, inclusive, have the property that some permutation of its digits is a multiple of 11 between 100 and 999? For example, both 121 and 211 have this property.

18. **(2019 AMC 10B #25)** How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

19. **(2020 AMC 10A #25)** Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

20. **(2023 AIME I #3)** A plane contains 40 lines, no 2 of which are parallel. Suppose that there are 3 points where exactly 3 lines intersect, 4 points where exactly 4 lines intersect, 5 points where exactly 5 lines intersect, 6 points where exactly 6 lines intersect, and no points where more than 6 lines intersect. Find the number of points where exactly 2 lines intersect.

21. **(2021 Fall AMC 12B #23)** What is the average number of pairs of consecutive integers in a randomly selected subset of 5 distinct integers chosen from the set $\{1, 2, 3, \ldots, 30\}$? (For example the set $\{1, 17, 18, 19, 30\}$ has 2 pairs of consecutive integers.)

22. **(2021 AMC 12A #23)** Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop-up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she ”wraps around” and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops ”up”, the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

23. **(2023 AMC 12B #23)** When n standard six-sided dice are rolled, the product of the numbers rolled can be any of 936 possible values. What is n?

24. **(2023 AMC 12A #24)** Let K be the number of sequences A_1, A_2, $\ldots$, A_n such that n is a positive integer less than or equal to 10, each A_i is a subset of $\{1, 2, 3, \ldots, 10\}$, and A_{i-1} is a subset of A_i for each i between 2 and n, inclusive. For example, $\{\}$, $\{5, 7\}$, $\{2, 5, 7\}$, $\{2, 5, 7\}$, $\{2, 5, 6, 7, 9\}$ is one such sequence, with $n = 5$.What is the remainder when K is divided by 10?

25. **(2017 AMc 12B #25)** A set of n people participate in an online video basketball tournament. Each person may be a member of any number of 5-player teams, but no two teams may have exactly the same 5 members. The site statistics show a curious fact: The average, over all subsets of size 9 of the set of n participants, of the number of complete teams whose members are among those 9 people is equal to the reciprocal of the average, over all subsets of size 8 of the set of n participants, of the number of complete teams whose members are among those 8 people. How many values n, $9 \leq n \leq 2017$, can be the number of participants?