Multiplicative Functions

ORMC

04/14/24

1 Multiplicative Functions

A function $f : \mathbb{N} \to \mathbb{N}$ is *multiplicative* when for every $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$, $f(mn) = f(m)f(n)$.

Problem 1.1. Suppose f is a multiplicative function, and you know how to calculate $f(p^k)$ whenever p is prime. How can you calculate $f(n)$ in general?

Problem 1.2 (Putnam 1963 A2)**.** Let $f : \mathbb{N} \to \mathbb{N}$ be a multiplicative function such that $f(2) = 2$ and f is strictly increasing, that is, for $m < n$, $f(m) < f(n)$.

Show that for all n, $f(n) = n$.

2 Sums of Divisors

Definition 2.1. For any positive integer n, let $\sigma(n) = \sum_{d|n} d$ be the sum of the divisors of n, including n.

Problem 2.2 (Putnam 1969 B1)**.** The positive integer n is divisible by 24. Show that the sum of all the positive divisors of $n - 1$ (including 1 and $n - 1$) is also divisible by 24.

Hint: Put divisors in pairs.

Problem 2.3. Calculate $\sigma(p^k)$ when p is prime.

Problem 2.4. Show that $\sigma(n)$ is multiplicative.

Problem 2.5 (BAMO 2019 Problem 5)**.** Every positive integer is either nice or naughty, and the Oracle of Numbers knows which are which. However, the Oracle will not directly tell you whether a number is nice or naughty. The only questions the Oracle will answer are questions of the form "What is the sum of all nice divisors of n?," where n is a number of the questioner's choice.

Show that for any given positive integer n less than 1 million, you can determine whether n is nice or naughty by asking the Oracle at most four questions.

3 Fermat's Little Theorem

Theorem 3.1 (Fermat's Little Theorem)**.** *Let p be a prime, and a an integer not divisible by p. Then*

$$a^{p-1} \equiv 1 \pmod{p}.$$

Problem 3.2. Prove Fermat's Little Theorem, starting by showing that the sets

$$\{1, 2, \ldots, p - 1\}, \{a, 2a, \ldots, (p - 1)a\}$$

contain the same numbers mod p.

Problem 3.3 (IMO 2005 Problem 4)**.** Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \ n \geq 1.$$

Hint: What would a_{-1} be? What does the sequence $a_1, a_2, a_3, \ldots$ act like mod a prime p?

Problem 3.4 (Putnam 2017 A1)**.** Let S be the smallest set of positive integers such that

- 2 is in S,
- n is in S whenever n^2 is in S,
- $(n+5)^2$ is in S whenever n is in S.

Which positive integers are not in S? (The set S is "smallest" in the sense that S is contained in any other such set.)

4 Euler's Totient Function

Definition 4.1. For any positive integer n, let $\phi(n)$ be the number of numbers k with $1 \leq k < n$ and $\gcd(k, n) = 1$. This is historically called *Euler's Totient Function.*

Problem 4.2. Calculate $\phi(p^k)$ for p prime.

Problem 4.3. Show that ϕ is multiplicative.

This function can be used to generalize Fermat's Little Theorem to general numbers, and not just primes:

Theorem 4.4 (Euler's Theorem)**.** *Let m be a positive integer, and a an integer with* $\gcd(a, m) = 1$. *Then*

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

Problem 4.5 (USAMO 1991 Problem 3)**.** Show that, for any fixed integer $n \geq 1$, the sequence

$$2, \ 2^2, \ 2^{2^2}, \ 2^{2^{2^2}}, \ldots \pmod{n}$$

is eventually constant.

5 A Bonus Problem

Problem 5.1 (Putnam 2021 A5)**.** Let A be the set of all integers n such that $1 \leq n \leq 2021$ and $\gcd(n, 2021) = 1$. For every nonnegative integer j, let $S(j) = \sum_{n \in A} n^j$. Determine all values of j such that $S(j)$ is a multiple of 2021.