ORMC AMC 10/12 Group
Week 2: Number Theory

April 14, 2024

1 Warm-Ups

1. **(2010 AMC 12A #1)** What is $(20 - (2010 - 201)) + (2010 - (201 - 20))$?

2. **(2019 AIME I #1)** Consider the integer

$$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99\ldots99}_{\text{321 digits}}.$$

   Find the sum of the digits of N.

3. **(2012 AIME II #2)** Two geometric sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_9.

4. **(2018 AMC 12B #3)** A line with slope 2 intersects a line with slope 6 at the point $(40, 30)$. What is the distance between the x-intercepts of these two lines?

5. **(1989 Putnam #A1)** Show that the only prime number in the sequence $101, 10101, 1010101, \ldots$ is 101.

2 Facts/Theorems

A lot of the exercises will require facts from various topics that we have discussed in previous classes. It would likely be helpful to go through the following list before starting the exercises, to make sure you remember what all of these are/ how to do them:

- Area-based Probability
- Modular arithmetic
- Binomial theorem
- Quadratic formula
- Graphing
- Common volume formulas (i.e. sphere, cone)
- Geometric and Arithmetic Sequences
- Logarithms
- Principle of Inclusion-Exclusion
- Complex roots to polynomials

One important fact that we haven't gone over in much detail is **Vieta's Formulas**. We know that a degree-n polynomial with real coefficients, i.e.

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

factors like:

$$p(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n),$$

where $r_1, \ldots, r_n$ are the (not necessarily distinct) roots of $p(x)$, and they are all real numbers, or come in complex conjugate pairs. The most commonly used parts of Vieta's formulas are for the constant term a_0 and the degree-$n-1$ coefficient a_{n-1}:

$$\begin{aligned} a_0 &= (-1)^n a_n \cdot r_1 r_2 \cdots r_n \\ a_{n-1} &= -a_n(-r_1 - r_2 - \cdots - r_n). \end{aligned}$$

That is, if we assume $a_n = 1$ (which we can always ensure by dividing through by a_n before factoring), then **the constant term is the product of the (negative) roots**, and **the degree-$n-1$ term is the (negative) sum of the roots**. In general, Vieta's formula says the following:

$$a_{n-k} = (-1)^k a_n \sum_{1 \le i_1 < \cdots < i_k \le n} r_{i_1} r_{i_2} \cdots r_{i_k},$$

which is easy to verify if we think about how distributing through the factorization works, and what all the coefficients of x^{n-k} terms would look like.

3 Exercises

1. **(2007 AMC 10A #10)** The Dunbar family consists of a mother, a father, and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?

2. **(2003 AMC 10A #12)** A point (x, y) is randomly picked from inside the rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 1)$, and $(0, 1)$. What is the probability that $x < y$?

3. **(2005 AMC 12A #10)** A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?

4. **(2003 AMC 12A #10)** Al, Bert, and Carl are the winners of a school drawing for a pile of Halloween candy, which they are to divide in a ratio of $3 : 2 : 1$, respectively. Due to some confusion they come at different times to claim their prizes, and each assumes he is the first to arrive. If each takes what he believes to be the correct share of candy, what fraction of the candy goes unclaimed?

5. **(2002 AMC 12A #10)** Sarah places four ounces of coffee into an eight-ounce cup and four ounces of cream into a second cup of the same size. She then pours half the coffee from the first cup to the second and, after stirring thoroughly, pours half the liquid in the second cup back to the first. What fraction of the liquid in the first cup is now cream?

6. **(2004 AMC 10A #13)** At a party, each man danced with exactly three women and each woman danced with exactly two men. Twelve men attended the party. How many women attended the party?

7. **(2006 AMC 12B #11)** Joe and JoAnn each bought 12 ounces of coffee in a 16-ounce cup. Joe drank 2 ounces of his coffee and then added 2 ounces of cream. JoAnn added 2 ounces of cream, stirred the coffee well, and then drank 2 ounces. What is the resulting ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee?

8. **(2002 AMC 12A #11)** Mr. Earl E. Bird gets up every day at 8:00 AM to go to work. If he drives at an average speed of 40 miles per hour, he will be late by 3 minutes. If he drives at an average speed of 60 miles per hour, he will be early by 3 minutes. How many miles per hour does Mr. Bird need to drive to get to work exactly on time?

9. **(2006 AMC 10B #14)** Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. What is q?

10. **(2003 AMC 12B #13)** An ice cream cone consists of a sphere of vanilla ice cream and a right circular cone that has the same diameter as the sphere. If the ice cream melts, it will exactly fill the cone. Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream. What is the ratio of the cone's height to its radius?

11. **(2008 AMC 10A #15)** Yesterday Han drove 1 hour longer than Ian at an average speed 5 miles per hour faster than Ian. Jan drove 2 hours longer than Ian at an average speed 10 miles per hour faster than Ian. Han drove 70 miles more than Ian. How many more miles did Jan drive than Ian?

12. **(2002 AMC 12A #12)** Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. The number of possible values of k is

13. **(2020 AMC 10A #16)** A point is chosen at random within the square in the coordinate plane whose vertices are $(0,0), (2020,0), (2020,2020)$, and $(0,2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x,y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

14. **(2004 AMC 12A #14)** A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term in the geometric progression?

15. **(2004 AMC 12A #15)** Brenda and Sally run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters?

16. **(2004 AMC 12A #16)** The set of all real numbers x for which

$$\log_{2004}(\log_{2003}(\log_{2002}(\log_{2001} x)))$$

is defined is $\{x \mid x > c\}$. What is the value of c?

17. **(2012 AMC 10A #19)** Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 P.M. How long, in minutes, was each day's lunch break?

18. **(2003 AMC 12B #17)** If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$?

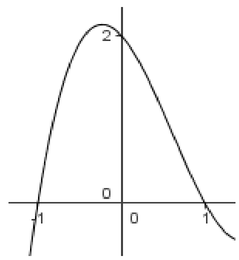
19. **(2007 AMC 10A #20)** Suppose that the number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$?

20. **(2018 AMC 10A #21)** Which of the following describes the set of values of a for which the curves $x^2 + y^2 = a^2$ and $y = x^2 - a$ in the real xy-plane intersect at exactly 3 points?

21. **(2007 AMC 12A #18)** The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2+i) = 0$. What is $a + b + c + d$?

22. **(2005 AMC 12A #19)** A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. If the odometer now reads 002005, how many miles has the car actually traveled?

23. **(2003 AMC 12B #20)** Part of the graph of $f(x) = ax^3 + bx^2 + cx + d$ is shown. What is b?



24. **(2011 AMC 10B #24)** A lattice point in an xy-coordinate system is any point (x, y) where both x and y are integers. The graph of $y = mx + 2$ passes through no lattice point with $0 < x \leq 100$ for all m such that $\frac{1}{2} < m < a$. What is the maximum possible value of a?

25. **(2000 AMC 12 #20)** If x, y, and z are positive numbers satisfying

$$x + \frac{1}{y} = 4, \qquad y + \frac{1}{z} = 1, \qquad \text{and} \qquad z + \frac{1}{x} = \frac{7}{3}$$

Then what is the value of xyz ?