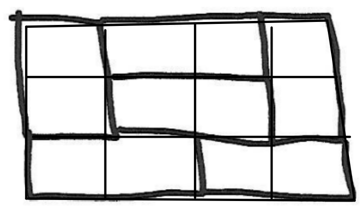
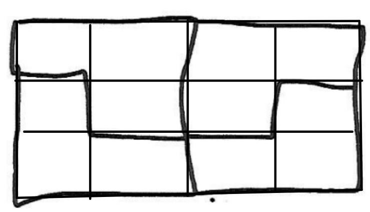
Induction

1 Warmup: Triomino's Pizza

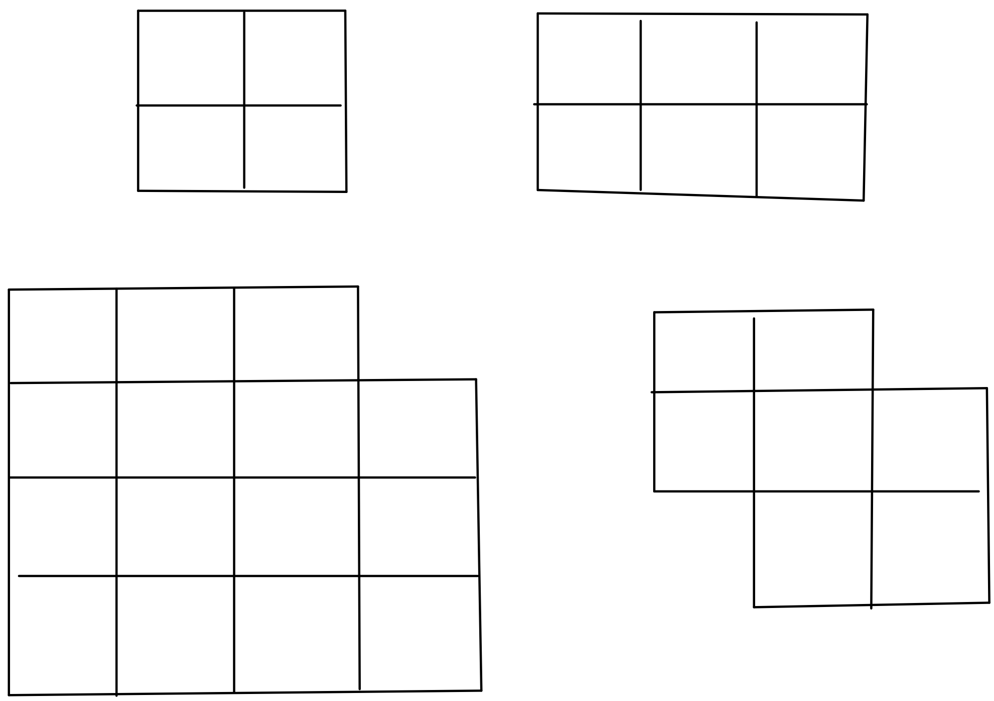
Domino's Pizza decides to implement a new company-wide policy: they can only cut their pizzas into "dominoes", or 2×1 rectangles, like shown:



This inspires a new competitor: Triomino's Pizza, who decide to only cut their pizzas into "triominoes", like shown:



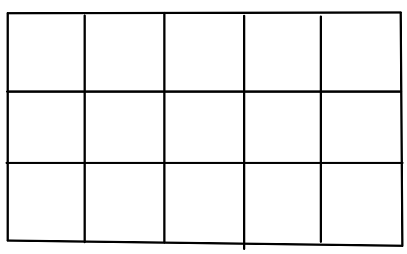
Problem 1.1. *Which of the following pizza's could be served by Domino's? What about by Triomino's? (Remember, a pizza can only be served by Domino's if it can be cut into Domino shaped slices; likewise, a pizza can only be served by Triomino's if it can be cut into Triomino sliced pieces.)*

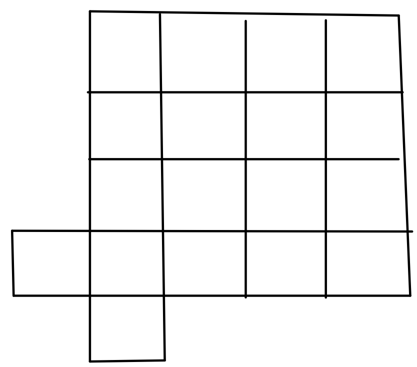


Problem 1.2. *Can you come up with a pizza shape that has area divisible by 3, but cannot be sold by Triomino's pizza?*

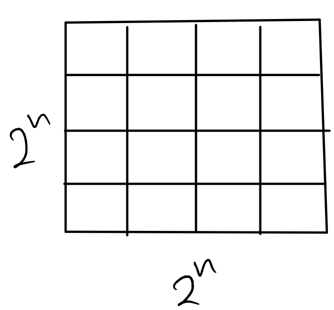
Problem 1.3. *Can you come up with a pizza whose area is divisible by 2, but cannot be sold by Domino's Pizza?*

Problem 1.4. *Check if these pizzas can be served by Domino's. What about Triomino's?*



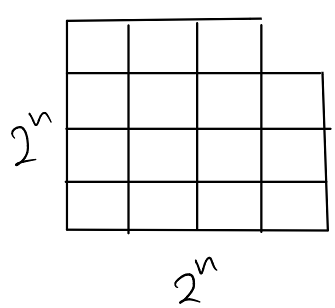


Problem 1.5. *To celebrate their 50th anniversary, Domino's releases a new line of pizza, called the "4^n pizzas". These pizzas measure 2^n by 2^n along a side. Show that the 4^1, 4^2 and 4^3 pizzas can be sold by Domino's. (Draw out all of the pizzas)*



Problem 1.6. *Assuming that the "4^{n-1} pizza" can be sold by Domino's, can you show that a "4^n pizza" can be served by Domino's? Use full sentences. (Hint: Break the "4^n pizza" into smaller pieces that you know can be served by Domino's)*

Problem 1.7. *Not to be outdone by Domino's, Triomino's releases the "$4^n - 1$ pizza", which looks just like the "4^n pizza" except it is missing a corner slice. Verify that the* $4^1 - 1$, $4^2 - 1$ *and* $4^3 - 1$ *pizzas can all be sold by Triomino's. (Draw out all of the pizzas)*



Problem 1.8. *Assuming that the "$4^{n-1} - 1$ pizza" can be sold by Triomino's, show that the "$4^n - 1$ pizza" can be served by Triomino's. Use full sentences. (Hint: Try doing what you did for the Domino's Pizza. Why doesn't it work? How can you fix it?)*

2 Induction

One of the most powerful proof techniques in mathematics is the method of induction. Induction works on the following principle:

- Let $P(n)$ be a logical statement that depends on the number n.
- Suppose we know $P(1)$ is true (or we prove as such)
- Suppose that whenever $P(k)$ is true, $P(k+1)$ is also true

Then, by the "principle of mathematical induction", $P(n)$ is true for every n.
Let us see some examples of induction in action.

Example 1. *Morgan likes pizza a lot. Morgan can always eat another slice of pizza– that is, if Morgan can eat* 5 *slices of pizza, he has no problem with eating a* 6^{th} *slice. Additionally, we know that Morgan has no problem eating a single slice of pizza. Show that Morgan can eat any number of pizza slices.*

The statement $P(n)$ *is then:*

- $P(n)$ = *Morgan can eat* n *slices of pizza*

We know additionally that

- $P(1)$ = *Morgan can eat* 1 *slice of pizza*

We also know that Morgan can always eat another slice of pizza (if he eats 1 slice, he can eat 2 slices. Therefore, if he eats k slices, he can also eat k+1 slices)

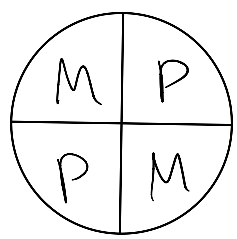
- *Now, suppose* $P(k)$ *is true. We know that* $P(k+1)$ *is also true by the above statement.*

- *Therefore, by the principle of mathematical induction, $P(n)$ is true for every n, i.e., Morgan can eat any number of pizza slices*

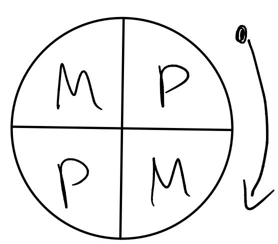
Let's see another induction problem:

Example 2. *Derek is about to eat a pizza. The pizza has an even number of slices, and half the slices have pepperoni, and the other half the slices have mushrooms. They are randomly assorted. Derek likes pepperoni, and dislikes mushrooms. The way Derek eats pizza is like this: He starts by eating a slice of pizza, and then eats the slice clockwise to that one, and then the slice clockwise to that one, and so on until the pizza is finished. Derek is okay with eating a slice with mushrooms, as long as at any given point he has eaten at least as many slices of pepperoni as slices with mushrooms. Why is it that Derek can happily eat his pizza?*

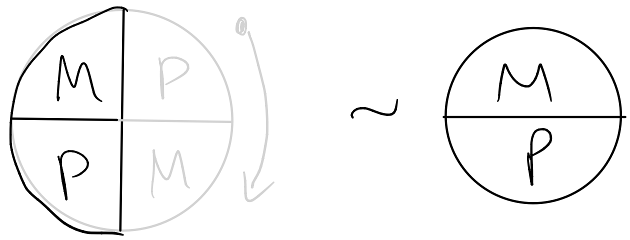
First, let's look at an example. Here is a pizza with four slices:



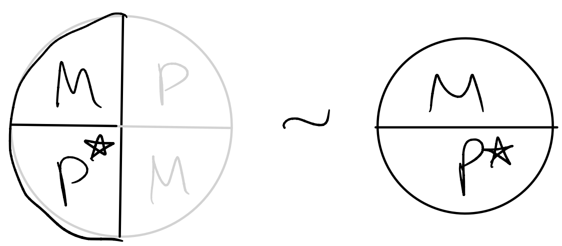
If Derek starts at the marked piece, then he will be able to eat his pizza happily in a clockwise direction:



Let's now prove that every pizza with 4 slices can be enjoyed by Derek.
Fact: there is always a mushroom slice that lies clockwise and next to a pepperoni slice.
If we remove these two adjacent slices, we get something that looks like a pizza with 2 slices, and clearly every pizza with 2 slices can be enjoyed by Derek:



Let's mark the slice where Derek must start in order to enjoy his 2-slice pizza:



Now what happens if we add the two slices back in? Because the mushroom slice lies clockwise of the pepperoni slice, Derek will eat the pepperoni slice first and then the mushroom slice. Derek can still enjoy his pizza with the two slices back in. Can we extend this process to bigger and bigger pizzas? Let us fit this in the language of induction:

Problem 2.1. *To solve this problem, we need to phrase it in a logical statement that depends on a number n.*

1. *What is the statement $P(n)$ that we are trying to prove?*

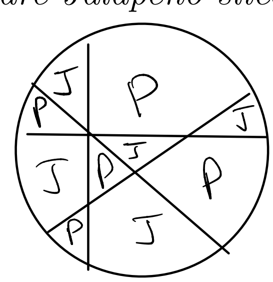
2. *The next part of induction is to show that the initial statement, $P(1)$ is true. Write the initial statement $P(1)$ in full sentences, and explain why it is true.*

3. *The next part of induction is to prove the "inductive step", that is: if $P(k)$ is true, then $P(k+1)$ is true.*
   Can you explain what the above statement is in full sentences for this problem?

4. *Explain why the statement from 3. is true in detailed sentences.*

Problem 2.2. *Round Table Pizza works with round pizzas, and cuts them with a pizza cutter in straight lines. Their specialty pizza is the "Pineapple Jalapeno Checker Pizza", where*

- *Every slice is either Pineapple or Jalapeno*
- *The only slices that share an edge with a Jalapeno slice are Pineapple slices, and the only slice that share an edge with a Pineapple slice are Jalapeno slices*



Show that no matter how Round Table Pizza cuts their pizzas, they can checker the Pineapple and Jalapeno slices:

1. *What is the inductive statement $P(n)$? (Hint: n will denote the number of cuts made.)*

2. *What is $P(1)$? Verify that it is true.*

3. *Why is it that if we know $P(k)$ is true, then we know that $P(k+1)$ is also true? (Hint: For starters, try showing that if $P(1)$ is true, then $P(2)$ is also true.)*

4. *Conclude that $P(n)$ is true for all n.*

Problem 2.3. *Show that:*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{1}$$

1. *Write out the statement $P(n)$*

2. *What does $P(1)$ say, and why is it true?*

3. *Prove that if $P(n)$ is true, then $P(n+1)$ is true.*

Problem 2.4. *Prove that the sum of the first n odd numbers is n^2*

Problem 2.5. *Prove that for any natural number n, the inequality $2^n > n^2$ holds when $n \geq 5$.*

Problem 2.6. *Prove that for any natural number n, the following holds: $3^n > 2^n + n$.*

Problem 2.7. *Prove by induction that for any integer $m > 0$ and any real numbers n, a,*

$$n + (n + a) + (n + 2a) + \cdots + (n + ma) = \frac{(m + 1)(2n + ma)}{2}.$$

Problem 2.8. *Prove that for any natural number n, following sum of squares formula holds:*

$$1^2 + 2^2 + ... + n^2 = \frac{n(n + 1)(2n + 1)}{6} \tag{2}$$

Problem 2.9. *Prove the formula for the sum of cubes:*

$$1^3 + 2^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{3}$$

Problem 2.10. *By now, you may have realised that induction is a powerful tool. However, we must be careful when applying it! Consider the following inductive proof to the statement "all horses are of the same color":*

- $P(n)$*: any group of* n *horses all have the same color.*
- $P(1)$*: any group of 1 horse all have the same color. This is clearly true.*
- *Suppose that* $P(k)$ *is true, that is, any group of* k *horses all have the same color. Consider a group of* $k+1$ *horses:* $h_1, h_2, \ldots, h_{k+1}$*. Since* $(h_1, h_2, \ldots, h_k)$ *is a group of* k *horses, they all have the same color. Similarly,* $(h_2, h_3, \ldots, h_{k+1})$ *is a group of* k *horses, so they all have the same color. This means* $h_1, h_2, \ldots h_{k+1}$ *all have the same color.*

- *Therefore, since* $P(1)$ *is true, and* $P(k+1)$ *is true whenever* $P(k)$ *is true, by the principle of mathematical induction,* $P(n)$ *is true for all* n*. That is, all horses are of the same color!*

Where is the mistake in the above argument?