ORMC AMC 10/12 Group
Week 2: Number Theory

April 7, 2024

1 Warm-Ups

1. **(1988 AJHSME #10)** Chris' birthday is on a Thursday this year. What day of the week will it be 60 days after her birthday?

2. **(1988 AHSME #12)** Each integer 1 through 9 is written on a separate slip of paper and all nine slips are put into a hat. Jack picks one of these slips at random and puts it back. Then Jill picks a slip at random. Which digit $(0-9)$ is most likely to be the units digit of the sum of Jack's integer and Jill's integer?

3. **(2015 AMC 10A #13)** Claudia has 12 coins, each of which is a 5-cent coin or a 10-cent coin. There are exactly 17 different values that can be obtained as combinations of one or more of her coins. How many 10-cent coins does Claudia have?

4. **(2018 AMC 10A #7)** For how many (not necessarily positive) integer values of n is the value of $4000 \cdot \left(\frac{2}{5}\right)^n$ an integer?

5. **(2020 AIME II #1)** Find the number of ordered pairs of positive integers (m, n) such that $m^2n = 20^{20}$.

6. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{a+b} = \frac{c}{c+d}$.

2 Facts/Theorems

A lot of the exercises will require number theory topics that we have discussed in previous classes. It would likely be helpful to go through the following list before starting the exercises, to make sure you remember what all of these are/ how to do them:

- Counting divisors
- Modular arithmetic rules
- LCM, and how to find it from prime factorization
- GCD, and how to find it from prime factorization
- Euclidean Algorithm
- Modular Inverses, how to find using Euclidean Alg.
- Difference of Squares
- Sum of Cubes factorization
- Fermat's Little Theorem
- Chinese Remainder Theorem

There are also a few more important facts for this worksheet, which you should have proven in exercises on previous worksheets, but which we haven't put much emphasis on up to this point:

1. **Bezout's Lemma:** Given two positive integers a, b, the smallest positive integer that can be written as $ma + nb$, where m, n are any integers (not necessarily positive), is $\gcd(a, b)$.

2. **Representing numbers in different bases:** Numbers in base 10 are generally written like

$$\underline{a}\ \underline{b}\ \underline{c}\ \underline{d},$$

where a, b, c, d are the digits of the number. What the above representation really means is

$$a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1 + d \cdot 10^0.$$

Thinking of numbers in this expansion is often useful, especially for converting to a different base B, where we would write the number in digit form like:

$$\underline{w}\ \underline{x}\ \underline{y}\ \underline{z}_B,$$

and its expanded form would be:

$$w \cdot B^3 + x \cdot B^2 + y \cdot B^1 + z \cdot B^0.$$

This expanded form gives us a better way to reason about different numbers when we're only given information about their digits, or when we need to convert/compare between different bases.

3. **Euler's Theorem:** An extension of Fermat's little theorem, we have that if $\gcd(a, m) = 1$, then

$$a^{\varphi}(m) \equiv 1 \pmod{m},$$

where $\varphi(m)$ is Euler's Totient function, and is equal to the number of positive integers less than m, which are relatively prime to m.

   (a) **Euler's Totient Function:** Recall that if m has a prime factorization:

$$m = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n},$$

then

$$\begin{aligned} \varphi(m) &= (p_1 - 1)p_1^{e_1 - 1}(p_2 - 1)p_2^{e_2 - 1} \cdots (p_n - 1)p_n^{e_n - 1} \\ &= m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_n}\right) \end{aligned}$$

3 Exercises

1. **(2020 AMC 8 #17)** How many positive integer factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely $1, 2, 3, 4, 6$, and 12.)

2. **(2022 AMC 10A #7)** The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n?

3. **(1990 AHSME #11)** How many positive integers less than 50 have an odd number of positive integer divisors?

4. **(2006 AMC 10B #11)** What is the tens digit in the sum $7! + 8! + 9! + ... + 2006!$

5. **(2005 AMC 10A #14)** How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

6. **(2005 AMC 10A #15)** How many positive cubes divide $3! \cdot 5! \cdot 7!$?

7. **(2003 AMC 10A #16)** What is the units digit of 13^{2003}?

8. **(2007 AMC 10A #17)** Suppose that m and n are positive integers such that $75m = n^3$. What is the minimum possible value of $m + n$?

9. **(2015 AMC 10A #18)** Hexadecimal (base-16) numbers are written using numeric digits 0 through 9 as well as the letters A through F to represent 10 through 15. Among the first 1000 positive integers, there are n whose hexadecimal representation contains only numeric digits. What is the sum of the digits of n?

10. **(2018 AMC 10A #19)** A number m is randomly selected from the set $\{11, 13, 15, 17, 19\}$, and a number n is randomly selected from $\{1999, 2000, 2001, \ldots, 2018\}$. What is the probability that m^n has a units digit of 1?

11. **(2003 AMC 10A #20)** How many base-10 three digit numbers n also have a three-digit base-9 representation and a three-digit base-11 representation?

12. **(2005 AMC 10A #21)** For how many positive integers n does $1 + 2 + ... + n$ evenly divide $6n$?

13. **(2005 AMC 10A #22)** Let S be the set of the 2005 smallest positive multiples of 4, and let T be the set of the 2005 smallest positive multiples of 6. How many elements are common to S and T?

14. **(2016 AMC 10A #22)** For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

15. **(2007 AMC 10A #23)** How many ordered pairs (m, n) of positive integers, with $m \geq n$, have the property that their squares differ by 96?

16. **(2005 AMC 10B #24)** Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers x and y satisfy $x^2 - y^2 = m^2$ for some positive integer m. What is $x + y + m$?

17. **(2006 AMC 10B #25)** Mr. Jones has eight children of different ages. On a family trip his oldest child, who is 9, spots a license plate with a 4-digit number in which each of two digits appears two times. "Look, daddy!" she exclaims. "That number is evenly divisible by the age of each of us kids!" "That's right," replies Mr. Jones, "and the last two digits just happen to be my age." Which of the following is not the age of one of Mr. Jones's children?

    (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

18. **(2002 AMC 12B #11)** The positive integers $A, B, A-B$, and $A+B$ are all prime numbers. The sum of these four primes is

(A) even (B) divisible by 3 (C) divisible by 5 (D) divisible by 7 (E) prime

19. **(2002 AMC 12B #12)** For how many integers n is $\dfrac{n}{20-n}$ the square of an integer?

20. **(2007 AMC 12A #12)** Integers a, b, c, and d, not necessarily distinct, are chosen independently and at random from 0 to 2007, inclusive. What is the probability that $ad - bc$ is even?

21. **(2003 AMC 12A #12)** Sally has five red cards numbered 1 through 5 and four blue cards numbered 3 through 6. She stacks the cards so that the colors alternate and so that the number on each red card divides evenly into the number on each neighboring blue card. What is the sum of the numbers on the middle three cards?

22. **(2006 AMC 12A #14)** Two farmers agree that pigs are worth 300 dollars and that goats are worth 210 dollars. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a 390 dollar debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

23. **(2011 AMC 12B #15)** How many positive two-digit integers are factors of $2^{24}-1$?

24. **(2009 AMC 12A #18)** For $k > 0$, let $I_k = 10\ldots064$, where there are k zeros between the 1 and the 6. Let $N(k)$ be the number of factors of 2 in the prime factorization of I_k. What is the maximum value of $N(k)$?

25. **(2016 AMC 12B #22)** For a certain positive integer n less than 1000, the decimal equivalent of $\frac{1}{n}$ is $0.\overline{abcdef}$, a repeating decimal of period of 6, and the decimal equivalent of $\frac{1}{n+6}$ is $0.\overline{wxyz}$, a repeating decimal of period 4. In which interval does n lie?

(A) $[1, 200]$ **(B)** $[201, 400]$ **(C)** $[401, 600]$ **(D)** $[601, 800]$ **(E)** $[801, 999]$

26. **(2010 AMC 12A #23)** The number obtained from the last two nonzero digits of $90!$ is equal to n. What is n?