Factorization 2

ORMC

04/07/24

1 Problems from *Problem Solving Through Problems*

Problem 1.1. Use unique factorization to prove that $\sqrt{2}$ is irrational.

Problem 1.2. Use unique factorization to prove that $\sqrt[3]{72}$ is irrational.

Problem 1.3. Prove that if ab, ac, and bc are perfect cubes for some positive integers a, b, c, then a, b, and c must also be perfect cubes.

Problem 1.4. A changing room has n lockers numbered 1 to n, and all are locked. A line of n attendants $P_1, P_2, \ldots, P_n$ file through the room in order. Each attendant P_k changes the condition of those lockers (and only those) whose numbers are divisible by k: if such a locker is unlocked, P_k will lock it; if it is locked, P_k will unlock it. Which lockers are unlocked after all n attendants have passed through the room?

Problem 1.5. The prime factorizations of $r+1$ positive integers (with $r > 1$) together involve only r primes. Prove that there is a subset of these integers whose product is a perfect square.

2 Problems from *Putnam and Beyond*

Problem 2.1 (Russian Mathematical Olympiad, 1995)**.** Is it possible to place 1995 different positive integers around a circle so that for any two adjacent numbers, the ratio of the greater to the smaller is a prime?

Problem 2.2. Does the answer in the previous problem change with 1995 replaced by 2024?

Problem 2.3. How many 0s are at the end of 25! ?

Problem 2.4. If x is a real number, then $\lfloor x \rfloor$ is the greatest integer with $\lfloor x \rfloor \leq x$.

Prove Polignac's Formula: If p is a prime number and n a positive integer, then the exponent of p in $n!$ is

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots$$

Problem 2.5. Find all positive integers n such that $n!$ ends in exactly 1000 zeros.

Problem 2.6. Let $n \geq 2$ be an integer. Prove that if k^2+k+n is a prime number for all $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is a prime number for all $0 \leq k \leq n - 2$.

Show that this works in particular for $n = 41$.

Hint: If this isn't true, then there is some smallest number s such that $s^2 + s + n$ is not prime, with $\sqrt{\frac{n}{3}} < s \leq n - 2$. Let p be the smallest prime dividing $s^2 + s + n$. How big can p be compared to s? Is s the smallest number k such that p divides $k^2 + k + n$?

Problem 2.7 (Russian Mathematical Olympiad, 1999)**.** Show that each positive integer can be written as the difference of two positive integers having the same number of distinct prime factors.

3 Competition Problems

Problem 3.1 (BAMO 2017 Problem C/1)**.** Find all natural numbers n such that when we multiply all divisors of n, we will obtain 10^9. Prove that your number(s) n works and that there are no other such numbers.

Problem 3.2 (BAMO 2016 Problem C/1)**.** The distinct prime factors of an integer are its prime factors listed without repetition. For example, the distinct prime factors of 40 are 2 and 5. Let $A = 2^k - 2$ and $B = 2^k \cdot A$, where k is an integer with $k > 1$. Show that, for every choice of k,

- A and B have the same set of distinct prime factors.
- $A + 1$ and $B + 1$ have the same set of distinct prime factors.

Problem 3.3 (USAMO 2005 Problem 1)**.** Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

Problem 3.4 (USAMO 2016 Problem 2)**.** Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

Problem 3.5 (Putnam 2020 A1)**.** How many positive integers N satisfy all of the following three conditions?

- N is divisible by 2020.
- N has at most 2020 decimal digits.
- The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

Problem 3.6 (Russian Mathematical Olympiad 2011 Problem 10.6)**.** Petya chose a natural number $a > 1$ and wrote fifteen numbers on the board: $1+a, 1+a^2, 1+a^3, \ldots, 1+a^{15}$. Then he erased some of the numbers so that any two remaining numbers are coprime. What is the maximum number of numbers that could remain on the board?

Problem 3.7 (Putnam 2009 B1)**.** Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2!5!}{3!3!3!}$$