ORMC AMC 10/12 Group
Week 1: Geometry

March 31, 2024

1 Warm-Ups

1. **(1951 AHSME #3)** If the length of a diagonal of a square is $a + b$, what is the area of the square?

2. What is the area of a triangle with side lengths $13, 14, 15$?

3. Triangle ABC has $\angle A = 45°$ and $AB = AC = 4$. What is BC^2?

4. **(1960 AHSME # 7)** Circle I passes through the center of, and is tangent to, circle II. The area of circle I is 4. What is the area of circle II?

5. Right triangle ABC with $m\angle B = 90°$ and $m\angle A = 30°$ is inscribed in a circle of radius 5. What is the arc length $\overset{\frown}{AB}$?

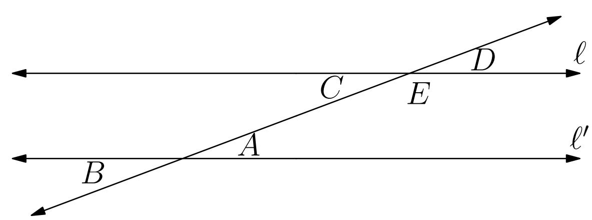
2 Facts/Theorems

A lot of the exercises will require geometry topics that we have discussed in previous classes. It would likely be helpful to go through the following list before starting the exercises, to make sure you remember what all of these are:

- Pythagorean theorem, common triples
- Area of circle (& sector), triangle, square
- Special triangles ($45-45-90$, $30-60-90$)
- Trig functions (sin, cos, tan)
- Power of a Point
- Law of cosines
- Inscribed angle theorem
- Stewart's Theorem
- Similar triangles (AA, SSS, ratios)
- Mass Points

There are also a few more important facts for this worksheet, which you should have proven in exercises on previous worksheets, but which we haven't put much emphasis on up to this point:

1. **Triangle Inequality:** If a, b, c are the side lengths of a triangle, then $a + b > c$
2. **Shoelace Theorem:** If a polygon P has vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, *listed in clockwise order*, then the area A of P is:
$$A = \frac{1}{2}\left|(x_1y_2 + x_2y_3 + \cdots + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_nx_1)\right|$$
3. **Parallels and Transversals:** In the following diagram, ℓ and ℓ' are parallel.



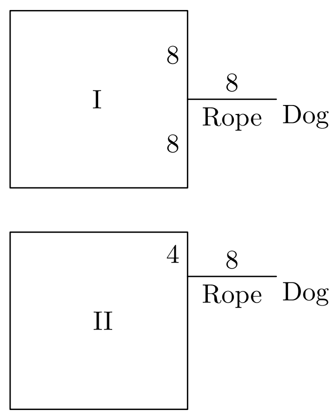
We have: $A \cong B \cong C \cong D$ and $A + E = 180°$. The pair (A, C) is called "alternate interior," (A, D) are called "corresponding," (B, D) are "alternate exterior," and (A, E) are "same-side interior."
4. **Perpendicular Slope:** If a line ℓ has slope m, then any perpendicular line has slope $\frac{-1}{m}$
5. **Distance Formula:** The distance between two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
6. **Ratios of Similar Figures:** If two similar figures have a side-length ratio of r, then the ratio of their (surface) areas is r^2, and the ratio of their volumes is r^3.
7. **Extended Law of Sines:** If triangle ABC has side lengths a, b, c, area K, and circumradius R, then:
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = 2R = \frac{abc}{2K}$$
8. **Heron's Formula:** If a triangle has side lengths a, b, c and semiperimeter $s = (a + b + c)/2$, then its area K is equal to:
$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

3 Exercises

1. **(2000 AMC 10 #10)** The sides of a triangle with positive area have lengths 4, 6, and x. The sides of a second triangle with positive area have lengths 4, 6, and y. What is the smallest positive number that is not a possible value of $|x - y|$?

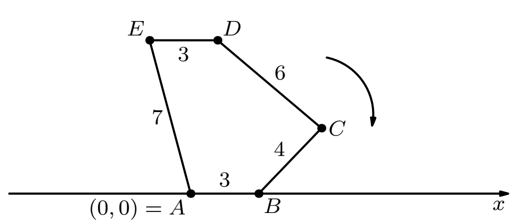
2. **(2007 AMC 10B #11)** A circle passes through the three vertices of an isosceles triangle that has two sides of length 3 and a base of length 2. What is the area of this circle?

3. **(2006 AMC 10A #12)** Rolly wishes to secure his dog with an 8-foot rope to a square shed that is 16 feet on each side. His preliminary drawings are shown.

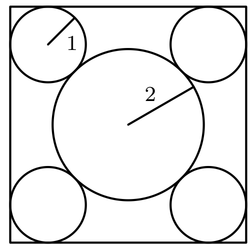


Which of these arrangements give the dog the greater area to roam, and by how many square feet?

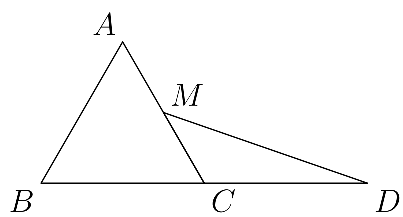
4. **(2009 AMC 10B #13)** As shown below, convex pentagon $ABCDE$ has sides $AB = 3$, $BC = 4$, $CD = 6$, $DE = 3$, and $EA = 7$. The pentagon is originally positioned in the plane with vertex A at the origin and vertex B on the positive x-axis. The pentagon is then rolled clockwise to the right along the x-axis. Which side will touch the point $x = 2009$ on the x-axis?



5. **(2007 AMC 10A #15)** Four circles of radius 1 are each tangent to two sides of a square and externally tangent to a circle of radius 2, as shown. What is the area of the square?

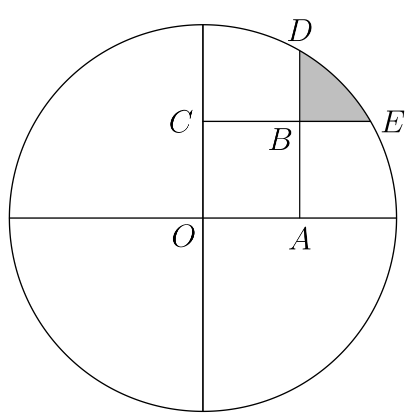


6. **(2005 AMC 10B #14)** Equilateral $\triangle ABC$ has side length 2, M is the midpoint of $\overline{AC}$, and C is the midpoint of $\overline{BD}$. What is the area of $\triangle CDM$?



7. **(2009 AMC 10A #17)** Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. Segment EF is constructed through B so that EF is perpendicular to DB, and A and C lie on DE and DF, respectively. What is EF?

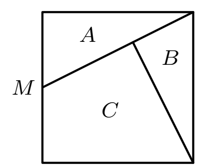
8. **(2006 AMC 10B #19)** A circle of radius 2 is centered at O. Square $OABC$ has side length 1. Sides AB and CB are extended past B to meet the circle at D and E, respectively. What is the area of the shaded region in the figure, which is bounded by BD, BE, and the minor arc connecting D and E?



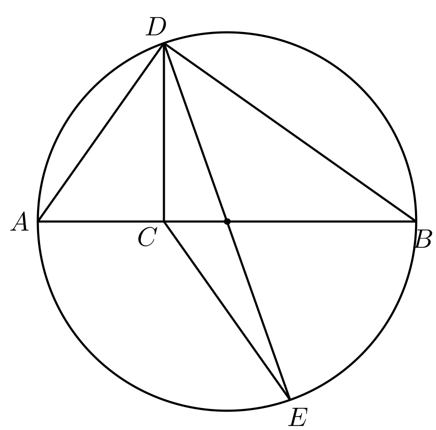
9. **(2006 AMC 10B #20)** In rectangle $ABCD$, we have $A = (6, -22)$, $B = (2006, 178)$, $D = (8, y)$, for some integer y. What is the area of rectangle $ABCD$?

10. **(2007 AMC 10A #21)** A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?

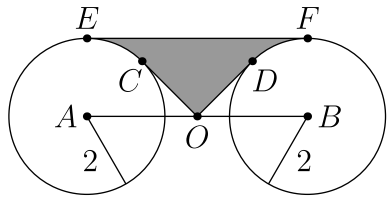
11. **(2009 AMC 10B #22)** A cubical cake with edge length 2 inches is iced on the sides and the top. It is cut vertically into three pieces as shown in this top view, where M is the midpoint of a top edge, and B is a right triangle. The piece whose top is triangle B contains c cubic inches of cake and s square inches of icing. What is $c + s$?



12. **(2005 AMC 10A #23)** Let AB be a diameter of a circle and let C be a point on AB with $2 \cdot AC = BC$. Let D and E be points on the circle such that $DC \perp AB$ and DE is a second diameter. What is the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$?



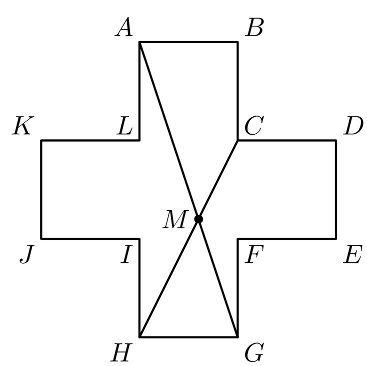
13. **(2007 AMC 10A #24)** Circles centered at A and B each have radius 2, as shown. Point O is the midpoint of $\overline{AB}$, and $OA = 2\sqrt{2}$. Segments OC and OD are tangent to the circles centered at A and B, respectively, and EF is a common tangent. What is the area of the shaded region $ECODF$?



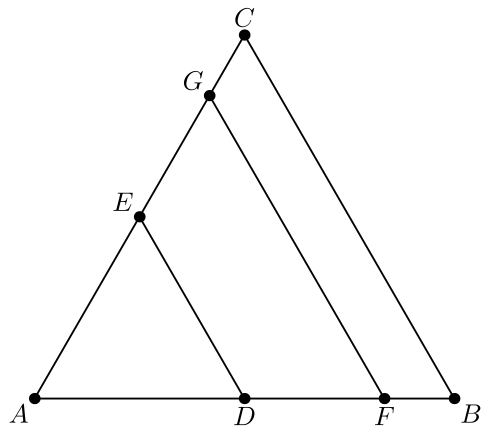
14. **(2014 AMC 12A #10)** Three congruent isosceles triangles are constructed with their bases on the sides of an equilateral triangle of side length 1. The sum of the areas of the three isosceles triangles is the same as the area of the equilateral triangle. What is the length of one of the two congruent sides of one of the isosceles triangles?

15. **(2009 AMC 10B #16)** Points A and C lie on a circle centered at O, each of $\overline{BA}$ and $\overline{BC}$ are tangent to the circle, and $\triangle ABC$ is equilateral. The circle intersects $\overline{BO}$ at D. What is $\frac{BD}{BO}$?

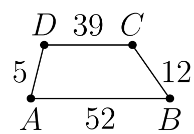
16. **(2007 AMC 10A #18)** Consider the 12-sided polygon $ABCDEFGHIJKL$, as shown. Each of its sides has length 4, and each two consecutive sides form a right angle. Suppose that $\overline{AG}$ and $\overline{CH}$ meet at M. What is the area of quadrilateral $ABCM$?



17. **(2013 AMC 12A #11)** Triangle ABC is equilateral with $AB = 1$. Points E and G are on $\overline{AC}$ and points D and F are on $\overline{AB}$ such that both $\overline{DE}$ and $\overline{FG}$ are parallel to $\overline{BC}$. Furthermore, triangle ADE and trapezoids $DFGE$ and $FBCG$ all have the same perimeter. What is $DE + FG$?



18. **(2002 AMC 10A #25)** In trapezoid $ABCD$ with bases AB and CD, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$. What is the area of $ABCD$?

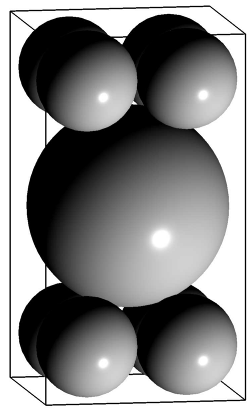


19. **(2013 AMC 12A #13)** Let points $A = (0, 0)$, $B = (1, 2)$, $C = (3, 3)$, and $D = (4, 0)$. Quadrilateral $ABCD$ is cut into equal area pieces by a line passing through A. This line intersects $\overline{CD}$ at point $\left(\frac{p}{q}, \frac{r}{s}\right)$, where these fractions are in lowest terms. What is $p + q + r + s$?

20. **(2020 AMC 12A #14)** Regular octagon $ABCDEFGH$ has area n. Let m be the area of quadrilateral $ACEG$. What is $\frac{m}{n}$?

21. **(2013 AMC 12A #12)** The angles in a particular triangle are in arithmetic progression, and the side lengths are $4, 5, x$. The sum of the possible values of x equals $a + \sqrt{b} + \sqrt{c}$ where a, b, and c are positive integers. What is $a + b + c$?

22. **(2014 AMC 12A #17)** A $4 \times 4 \times h$ rectangular box contains a sphere of radius 2 and eight smaller spheres of radius 1. The smaller spheres are each tangent to three sides of the box, and the larger sphere is tangent to each of the smaller spheres. What is h?

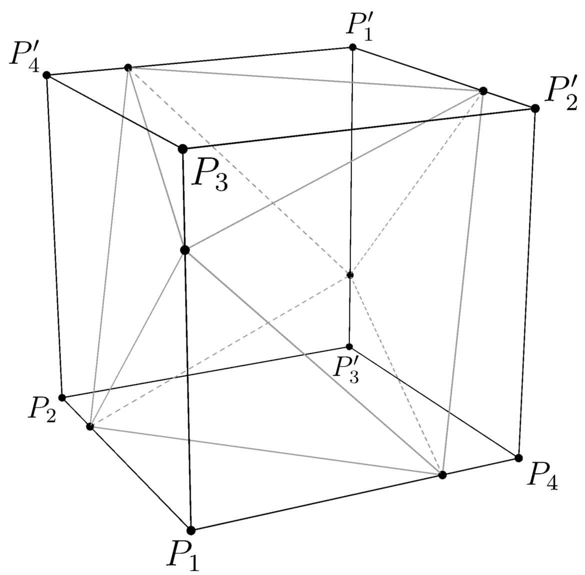


23. **(2017 AMC 12B #15)** Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

24. **(2012 AMC 12A #16)** Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

25. **(2012 AMC 12A #18)** Triangle ABC has $AB = 27$, $AC = 26$, and $BC = 25$. Let I be the intersection of the internal angle bisectors of $\triangle ABC$. What is BI?

26. **(2012 AMC 12B #19)** A unit cube has vertices $P_1, P_2, P_3, P_4, P_1', P_2', P_3'$, and P_4'. Vertices P_2, P_3, and P_4 are adjacent to P_1, and for $1 \le i \le 4$, vertices P_i and P_i' are opposite to each other. A regular octahedron has one vertex in each of the segments P_1P_2, P_1P_3, P_1P_4, $P_1'P_2'$, $P_1'P_3'$, and $P_1'P_4'$. What is the octahedron's side length?



27. **(2012 AMC 12B #20)** A trapezoid has side lengths 3, 5, 7, and 11. The sum of all the possible areas of the trapezoid can be written in the form of $r_1\sqrt{n_1} + r_2\sqrt{n_2} + r_3$, where r_1, r_2, and r_3 are rational numbers and n_1 and n_2 are positive integers not divisible by the square of any prime. What is the greatest integer less than or equal to $r_1 + r_2 + r_3 + n_1 + n_2$?