Divisibility II

Math Circle (Intermediate)

November 8, 2012

1. **DEFINITION: Two numbers are COPRIME or RELATIVELY PRIME if they have no common divisors greater than 1.**
   Note that any pair of prime numbers are relative prime, and that 1 is relatively prime to any natural number.

   On last week's handout, we proved that if a natural number n is divisible by 4 and by 3, then it must be divisible by $4 \cdot 3 = 12$. However, we disproved a very similar-looking statement: If a natural number n is divisible by 4 and by 3, then it must be divisible by $4 \cdot 6 = 24$.

   (a) Can you spot the key difference that might cause one of the above statements to be true while keeping the other false?

(b) Let's see if we can find another example.

i. If a natural number n is divisible by 4 and 5, is it also divisible by $4 \cdot 5 = 20$? Prove it or find a counterexample!

ii. If a natural number n is divisible by 4 and 6, is it also divisible by $4 \cdot 6 = 24$? Prove it or find a counterexample!

(c) Use your results from parts (a) and (b) to form a generalized hypothesis about the divisibility of a natural number by the product of two smaller natural numbers.
(Hint: Your hypothesis will have the form "If some natural number is... then it is...")

(d) Prove it!

2. **Prove or disprove:** If the natural number pA is divisible by q, where p and q are relatively prime, then A is also divisible by q.

3. Given two different prime numbers p and q, find the number of different divisors of the number:

   (a) pq

   (b) p^2q

(c) p^2q^2

(d) p^nq^m

4. **Prove or disprove**: The product of any three consecutive natural numbers is divisible by six.

5. Find the smallest natural number n such that $n!$ is divisible by 990.

6. For some natural number n, can the number $n!$ have exactly five zeros at the end of its decimal representation? **Prove or disprove it!**

7. Find all solutions in the natural numbers to the equation $x^2 - y^2 = 31$.

8. **Prove or disprove**: Any two natural numbers a and b must satisfy the equation
$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab.$$

9. **Prove or disprove**: If a natural number N has an odd number of divisors, then N must be a perfect square.

More problems!

10. Find the last digit of the number 1989^{1989}.

11. Find the last digit of the numebr 2^{50}.

12. Find the last digit of 777^{777}.

13. Find the remainder when 2^{100} is divided by 3.

14. Find the remainder when 3^{1989} is divided by 7.

15. **Prove or disprove:** The sum $2222^{5555} + 5555^{2222}$ is divisible by seven (that is, the remainder of the given number when divided by 7 is zero).

16. Find the last digit of the number 7^{7^7}.

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"