Summer Session Week 8

Questions

August 2021

Rules are easy:

You have 2 attempts for problem. If you solve the problem with your first attempt – you get 10 points. If you solve it with your second attempt – you get 5 points.

1 Problems

1. 12 people are taking part in a chess tournament. Each participant plays every other participant exactly once. Contestants earn 1 point for each victory, 0.5 points for each draw, and 0 points for each defeat. To earn the title "Master of Chess", a contestant must score more than 70% of the theoretical maximum number of points (had they won all games). What is the largest number of participants who can receive the "Master of Chess" title?

2. There are 4 pillars in the corners of the square pool. How can the pool be expanded so that the pillars remain on land, the area of the pool doubles, and the shape remains square?

3. If $f : \mathbb{R} \to \mathbb{R}$, then the kernel of f is $\ker(f) = \{x \in \mathbb{R} | f(x) = 0\}$. Find $\ker(f)$ if $f(x) = 7\left(x + \frac{1}{x}\right) - 2\left(x^2 + \frac{1}{x^2}\right) - 9$.

4. Let A be a set that follows two rules:
   1. If $a \in A$, then $a + a \in A$
   2. If $a \in A$, then $\frac{1}{a} \in A$

   Suppose that we know $1 \in A$. What is the biggest set we also know is in A?

5. If two sets A and B have 99 elements in common, then how many elements are shared by the sets $A \times B$ and $B \times A$?

6. Find a square that, if five is subtracted or added to it, the result will be another square.

7. An elementary school teacher in New York state had her purse stolen. The thief had to be Lilian, Judy, David, Theo, or Margaret. When questioned, each child made three statements:
   Lilian:
   (1) I didn't take the purse.
   (2) I have never in my life stolen anything.
   (3) Theo did it.
   Judy:
   (4) I didn't take the purse.
   (5) My daddy is rich enough, and I have a purse of my own.
   (6) Margaret knows who did it.
   David:
   (7) I didn't take the purse.
   (8) I didn't know Margaret before I enrolled in this school.
   (9) Theo did it.
   Theo:
   (10) I am not guilty.
   (11) Margaret did it.
   (12) Lillian is lying when she says I stole the purse.
   Margaret:
   (13) I didn't take the teacher's purse.
   (14) Judy is guilty.
   (15) David can vouch for me because he has known me since I was born.
   Later, each child admitted that two of his statements were true and one was false. Assuming this is true, who stole the purse?

8. Find two functions f and g such that $(fog)^{-1}\{1, 8, 27, 125\} = \{-2, -1, 0, 2\}$.

9. How many bijections are there from the set X to itself given $|X| = n, n \in \mathbb{N}$? (The answer should include n)

10. Let $\Omega = \{1, 2, 3, 4\}$ be a set, and let $\mathcal{B}$ be a collection of subsets of Ω. There are 2 rules about the set $\mathcal{B}$:

    a) Suppose $A \subset \Omega$. If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$ (A^c denotes the complement of A);

    b) Suppose $A_1, A_2 \subset \Omega$. If $A_1, A_2 \in \mathcal{B}$, then $A_1 \cup A_2 \in \mathcal{B}$.

    Now given $\{1, 2\}, \{2, 3\}, \{1, 3\}, \{4\} \in \mathcal{B}$, what is the cardinality of $\mathcal{B}$?

11. One hundred passengers board a plane with exactly 100 seats. The first passenger takes a seat at random. The second passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. The third passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. This process continues until all the 100 passengers have boarded the plane. What is the probability that the last passenger takes his own seat?

12. Randomly choose 3 points on the circle. What is the probability that the origin of the circle is inside the triangle formed by these 3 points?

13. Consider a standard 8×8 chessboard with two opposite corners removed. How many ways are there to tile the board with exactly 31 dominoes, where each domino covers exactly two adjacent squares?

14. Refer to Problem 3 for the definition of the kernel of a function. Let $f : \mathbb{R} \to \mathbb{R}$ be an injective function that satisfies $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. What is the kernel of f?

15. What is the cardinality of $\mathbb{Q} \times \mathbb{Q}$? (Hint: Review Exercises 17, 24, and 25 from the last worksheet).

16. Freddy the frog has just left his frog office and wants to return to his frog home, which is 15 meters away. At any point in time, Freddy can either take a short hop, which will bring him 1 meter closer to his home, or a large leap, which will bring him 3 meters closer. How many different sequences of short hops and large leaps can Freddy take to get home?

17. For how many ordered pairs of integers (x, y) satisfy the equation
$$x^{2020} + y^2 = 2y?$$

18. What is the value of
$$1 + 2 + 3 - 4 + 5 + 6 + 7 - 8 + \cdots + 197 + 198 + 199 - 200?$$

19. The function f is defined by
$$f(x) = \lfloor |x| \rfloor - |\lfloor x \rfloor|$$
for all real numbers x, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to the real number r. What is the range of f?

20. Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?