Exponents and Logarithms

1 Warm-up

Problem 1.1. *Once Mary ate a half of the peaches from a can, the level of the syrup decreased by one third. How much would decrease the syrup level from the new one, if Mary eats a half of the peaches left?*

2 Exponents

Let $b > 0$. The following are the fundamental laws of exponential functions:

$$b^x \times b^y = b^{x+y} \tag{1}$$

$$b^0 = 1 \tag{2}$$

$$b^{-x} = \frac{1}{b^x} \tag{3}$$

$$\frac{b^x}{b^y} = b^{x-y} \tag{4}$$

$$(b^x)^y = b^{xy} \tag{5}$$

$$\sqrt[n]{b} = b^{\frac{1}{n}} \tag{6}$$

$$\sqrt[n]{b^m} = (\sqrt[n]{b})^m = b^{\frac{m}{n}} \tag{7}$$

Problem 2.1. *Use the laws of the exponents to represent the following as powers of two.*

- $2^5 \times 8^2 =$

- $\dfrac{2^5 \times 4^3}{16^2} =$

Problem 2.2. *Use the laws of the exponents to simplify the following as much as you can.*

- $(a^x)^2 \times a^y \times a^{-2x} =$

- $\dfrac{(a^{-x})^{-y} \times a^3}{a^{xy-3}} =$

Problem 2.3. *Derive property 2 of exponential functions from property 1.*

Problem 2.4. *Derive property 3 from properties 1 and 2.*

Problem 2.5. *Derive property 4 from properties 1 and 3.*

Problem 2.6. *Derive property 6 from property 5.*

Problem 2.7. *Derive property 7 from properties 5 and 6.*

Problem 2.8. *To what power should one raise 10 to get 10,000,000?*

Problem 2.9. *To what power should one raise 100 to get 10,000,000?*

Problem 2.10. *To what power should one raise 0.5 to get 256?*

Problem 2.11. *To what power should one raise* $\frac{1}{9}$ *to get 3?*

Problem 2.12. *Solve the following equation.*

$$3^{12x} = \left(\frac{1}{9}\right)^3$$

3 Logarithms

The function $\log_b x$, called *logarithm to base b of x*, answers the following question: to what power should I raise the base b to get x? For example, $\log_2 8 = 3$.

Problem 3.1. *Find the following logarithms.*

- $\log_2 1024 =$

- $\log_{\frac{1}{2}} 1024 =$

- $\log_{10} 1000 =$

- $\log_{10} 0.001 =$

Problem 3.2. *Use the definition of the logarithm to prove that* $b^{\log_b x} = x$ *and* $\log_b b^y = y$.

Since logarithmic functions are inverses of the exponential ones, properties of the logarithms mirror the formulas 1 - 5.

$$
\begin{aligned}
&b^x \times b^y = b^{x+y} && \log_b(pq) = \log_b p + \log_b q \\
&b^0 = 1 && \log_b 1 = 0 \\
&b^{-x} = \frac{1}{b^x} && \log_b \frac{1}{p} = -\log_b p \\
&\frac{b^x}{b^y} = b^{x-y} && \log_b\left(\frac{p}{q}\right) = \log_b p - \log_b q \\
&(b^x)^y = b^{xy} && \log_b p^y = y \log_b p
\end{aligned}
$$

Problem 3.3. *Use 1 to prove the following formula.*

$$\log_b(pq) = \log_b p + \log_b q \tag{8}$$

Problem 3.4. *Use 2 to prove the following formula.*

$$\log_b 1 = 0 \tag{9}$$

Problem 3.5. *Use 3 to prove the following formula.*

$$\log_b\left(\frac{1}{p}\right) = -\log_b p \tag{10}$$

Problem 3.6. *Prove the following formula:*

$$\log_b x = \left(\frac{lnx}{lnb}\right) \tag{11}$$

Problem 3.7. *Use the formula above to find the following values with a calculator.*

- $\log_2 100 =$

- $\log_5 21 =$

Another logarithm, less important, but about as widespread as the natural one, is the *common logarithm*, $\lg x = log_{10}x$ The formula for which:

$$\log_b x = \left(\frac{lgx}{lgb}\right) \tag{12}$$

is proven similar to previous formula. Solve to following problems by using this formula accordingly:

Problem 3.8. *Use **Formula (12)** to find the following values with a calculator.*

- $\log_3 17 =$

- $log_5 e =$

Problem 3.9. *Prove the general formula for a base switch below.*

$$log_b x = \frac{log_c x}{log_c b} \tag{13}$$

Problem 3.10. *Find the following values:*

- $\log_{\sqrt{3}} 81 =$

- $\log_6 3 + log_6 2 =$

- $log_2 2\sqrt{2} =$

- $2\log_4 2 =$

Harder Problems:

Problem 3.11.

- *Given* $log_2 7a = b$, *find* $log_{\sqrt{3}a^6}$

- *How is* $\log_b a$ *related to* $log_a b$*?*

- *Given* $\log_7 = a$, *find* $log_{1/2} 28$

- *Given* $\log_7 2$, *find* $\log_{1/2} 28$

Here are a couple of fun problems with real-life applications:

Problem 3.12. ***Compound Interest Calculation:***

Interest *is the monetary charge of borrowing money, typically expressed as an annual percentage rate. In simpler terms, interest is like a reward the bank gives you for trusting them to look after your money. The more money you have in your account, and the longer you keep it there, the more interest you can earn.*

The bank calculates interest as a percentage of the total amount in a bank account. For example, if the bank pays 1% *interest and if there is \$500 in your account, then you will earn \$5 in interest over a year. It may not seem like a lot, but the great thing about interest is that it builds on itself. For example, if you start with \$500 in your account and you earn \$5 in interest over a year, you now have \$505 in your account. The next year, the bank will calculate your interest based on that new, higher amount. The interest you gain each year will continue to grow. (Khan Academy)*

Problem 1 : *The current rate of interest quoted by a bank on its time-deposit account is 4% per year.* ***Time-deposit*** *is a type of savings account where you agree to keep your money with the bank for a certain amount of time (a couple of months to a couple of years). In return, the bank will give you a higher* *interest rate. John opens a time-deposit savings account with a deposit of 1000$. Find the amount of account balance at the end of 2 years, and assume no change to principal money (Initial deposit into account).*

Effective Rates of Interest: *In practice, interest may be credited or charged more frequently than once per year. Many bank accounts pay interest monthly. If that is the case, simply divide the annual interest rate to the period that is charged (if monthly, divide by 12) and compound by the equivalent period amount (if monthly, 12 * year).*

Problem 2: *The current rate of interest quoted by a bank on its time-deposit account is 4% per year. John opens a time-deposit savings account with a deposit of 1000$. Find the amount of account balance at the end of 2 years, and assume no change to principal money (Initial deposit into account). This time, interest is compounded every half year, so your compound period is every 6 months. (Hint: year/2)*

Problem 3: *Compare results of Example 1 and Example 2. Notice the difference? Figure out why. Then, come up with the general formula for interest rate calculations.*