Barycentric Coordinates

ORMC

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1 Vectors

In order to best understand barycentric coordinates and related techniques, we have to understand how to think of points in the plane as *vectors*. In practice, this just means we can "add" two points together by adding each of their coordinates, and "scale" a point by a real number a by multiplying each coordinate by a. By a "vector expression", I will mean something we can get by starting with some vector variables, scaling vectors by real numbers, and adding vectors.

Problem 1.1. Given points A, B in the plane, find a vector expression for the midpoint of AB.

Problem 1.2. Suppose A, B, C are points on a common line, with C dividing the line segment AB such that the ratio $|AC| : |BC| = m : n$. Find a vector expression for C in terms of A and B.

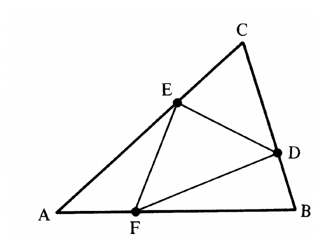
2 Centroids

A *median* of a triangle is the line from one vertex to the midpoint of the opposite side. The *centroid* is the intersection of all three medians.

Problem 2.1. Let ABC be a triangle, let D be the midpoint of BC, and let E be the centroid of ABC. What's the ratio of AE to AD?

Problem 2.2. Given a triangle ABC, find a vector expression for the centroid in terms of A, B, C.

Problem 2.3 (Problem Solving Through Problems)**.** In a triangle ABC the points D, E, F trisect the sides so that $BC = 3BD$, $CA = 3CE$, and $AB = 3AF$. Show that triangles ABC and DEF have the same centroid.



3 Barycentric Coordinates

Beyond just referring to a point in the plane with the vector (x, y), sometimes other coordinate systems are useful. For triangle-based geometry problems, often the most useful system of coordinates are *barycentric* coordinates, which is a coordinate system based on the corners of a specific triangle.

Suppose ABC is a triangle in the plane. Any point P in the plane can be written uniquely as $p = xA+yB+zC$, where x, y, z are real numbers with $x+y+z = 1$. We can denote such a point as (x, y, z), and if x', y', z' are real numbers with the right ratio, that is, $p = \frac{x'}{x'+y'+z'}A + \frac{y'}{x'+y'+z'}B + \frac{z'}{x'+y'+z'}C$, then we say that $(x' : y' : z')$ is P. Just remember if you see that notation that you can divide through by $x' + y' + z'$ to get the proper barycentric coordinates.

Problem 3.1. What are the barycentric coordinates of the centroid of ABC?

3.1 Lines

Lines in barycentric coordinates are given by the equations $ux+vy+wz = c$ when u, v, w, c are real, and not all of u, v, w are zero.

Problem 3.2. For any u, v, w, let's check that $ux + vy + wz = c$ actually defines a line.

First check that if P, Q are points in this set, and R is a point on the line PQ, that R also satisfies the equation. (Hint: For each such R, there is some $t \in \mathbb{R}$ with $r = tp + (1 - t)q$.)

This shows that every line connecting P and Q in the set is also in the set, meaning that the set is either a line or the whole plane. Show that if the set is the whole plane, that $u = v = w = c = 0$, so if any of them is nonzero, we only get a line.

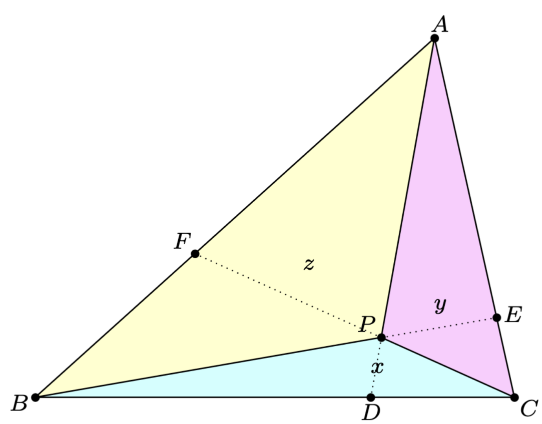
Problem 3.3. Show that every line can be given by an equation of the form $ux + vy + wz = 0$ (we're getting rid of c).

Problem 3.4. For which values of u, v, w does $ux + vy + wz = 0$ define a line through A?

Problem 3.5. Let $h_A(P)$ be the height of a point above the line BC, defined so that $\frac{1}{2}h(A)|BC|$ is the area of ABC, and $h(B) = h(C) = 0$. Check that for every h, $h_A(P) = h$ defines a line, and find an equation for this line in barycentric coordinates.

3.2 Areas

We can also interpret barycentric coordinates as areas. Split the triangle ABC into three parts as in this picture, thanks to Zach Abel:



Problem 3.6. Show that if P is given by barycentric coordinates (x, y, z), and the area of ABC is V, then the areas of the three subtriangles in the figure are (xV, yV, zV).

Hint: Find a barycentric coordinates formula for the height of the point P above the line BC.

3.3 Circles

The *incircle* of a triangle is the largest circle that can be placed inside the triangle - it is tangent to each side of the triangle at a unique point, and its center is called the *incenter*.

Problem 3.7. Find the barycentric coordinates of the incenter of ABC in terms of the sidelengths $a = |BC|, b = |AC|, c = |AB|$ of the triangle.

4 Problems

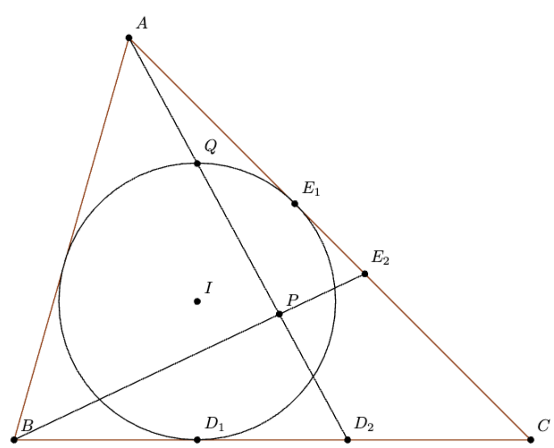
4.1 Theorems

Problem 4.1 (Ceva's Theorem)**.** Let ABC be a triangle, let D lie on BC, let E lie on AC, and let F lie on AB. Show that AD, BE, and CF intersect at a common point if and only if $\frac{|BD||CE||AF|}{|DC||EA||FB|} = 1$.

4.2 2001 USAMO Problem 2

First, I'll give a USAMO problem, and then I'll give hints as to solve it, following a solution by Evan Chen (whose image I am also using). Don't get stuck trying to solve it without looking at the hints first!

Problem 4.2 (2001 USAMO Problem 2)**.** Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.



Problem 4.3. Let F_1 be the point where ω is tangent to AB. Then (probably not using barycentric coordinates) show that $|AF_1| = |AE_1|$, $|BD_1| = |BF_1|$, and $|CD_1| = |CE_1|$. Use this, and the notation $s = \frac{a+b+c}{2}$ (the s stands for *semiperimeter*) to solve for $|BD_1|$ and $|CD_1|$ in terms of s, a, b, c.

Problem 4.4. Find barycentric coordinates for D_1, D_2, E_1, E_2, and P.

Problem 4.5. Find the barycentric coordinates for the point Q' on the line AD_2 with $|AQ'| = |PD_2|$.

Hint: The midpoint of PQ is the same as the midpoint of AD_2.

Problem 4.6. Find the midpoint of Q and D_1, and finish the problem.

4.3 1992 Putnam A6

Problem 4.7. Every point in the plane has barycentric coordinates. Which points (x, y, z) given in barycentric coordinates actually lie inside the triangle ABC?

Problem 4.8. Given a point P on a circle or sphere, its *antipode*, which we will denote P', is the point such that the center of the circle or sphere is the midpoint of PP'. If triangle ABC is inscribed in a circle with center O, and O has barycentric coordinates (x, y, z), what are its barycentric coordinates in the triangle $A'BC$?

Barycentric coordinates work for tetrahedra also, but you need 4 coordinates! We can apply that to this Putnam problem.

Problem 4.9. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is in- dependently chosen relative to a uniform distribution on the sphere.)

4.4 2008 USAMO Problem 2

You can solve the following USAMO problem with the following equations. If you are interested in proving them, ask me for hints, or look up Evan Chen's notes on barycentric coordinates after class.

The perpendicular bisector of BC is given by the equation

$$a^2(z - y) + (c^2 - b^2)x = 0$$

and the circumcircle (the smallest circle containing ABC) is given by the equation

$$a^2yz + b^2zx + c^2xy = 0.$$

Now here's the problem:

Problem 4.10 (2008 USAMO Problem 2). Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.