Approximating Numbers Part I: Continued Fractions

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1 Decimal Approximations

Problem 1 *Convert the following decimal expressions into fractions. (Recall that a bar over digits means those digits are repeating.)*

- 0.2

  Solution: $1/5$

- $1.\overline{3}$

  Solution: $4/3$

- $0.1\overline{7}$

  Solution: $16/90$ *or* $8/45$

- $3.14\overline{15}$

  Solution: $31101/9900$ *or* $10367/3300$

Problem 2 *Convert the following fractions into decimal expressions.*

- $\frac{1}{4}$

  Solution: 0.25

- $\frac{16}{9}$

  Solution: $1.\overline{7}$

- $\frac{35}{33}$

  Solution: $1.\overline{02}$

- $\frac{17}{12}$

  Solution: $1.41\overline{6}$

The examples on the first page are all of *rational numbers*—that is, numbers that can be written as a fraction a/b of two integers a and b. Not all real numbers are rational, a fact that we will see as soon as we have a way to test whether a number is rational. We'll first prove the following theorem:

Theorem 1 *A number is rational if and only if its decimal expansion is either finite or eventually repeating.*

To prove this theorem, we'll have to be able to add repeating digits, as we have seen previously. As with many infinite repeating patterns, we use the following trick to evaluate such expressions.

Problem 3 *Evaluate*

$$S = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots$$

by finding another S in the right-hand side. (Hint: Try rewriting $2 + 1 + 1/2 + 1/4 + \dots$ in terms of S.)

Solution: Since

$$\frac{1}{2}S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots$$

we have

$$S = 4 + \frac{1}{2}S \Rightarrow S = 8$$

Problem 4 *Evaluate*

$$S = 10 + 1 + \frac{1}{10} + \frac{1}{100} + \dots$$

by finding another S in the right-hand side.

Solution: Since

$$\frac{1}{10}S = 1 + \frac{1}{10} + \frac{1}{100} + \dots$$

we have

$$S = 10 + \frac{1}{10}S \Rightarrow S = \frac{100}{9}$$

In general, infinite sums like these examples are called *geometric series.* Geometric series are determined by their first term and their *ratio*, which is the quotient of any two successive terms.

Problem 5 *Prove the geometric series formula, which states that for any real numbers a and $-1 < r < 1$*[1]*,*

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r}$$

Solution: Let

$$S = a + ar + ar^2 + ar^3 + \dots$$

Note that

$$rS = ar + ar^2 + ar^3 + \dots$$

so that

$$S = a + rS \Rightarrow S = \frac{a}{1-r}$$

[1] $|r| < 1$ is a technical condition needed for the sum on the left-hand side to make sense mathematically. You will learn more about this when you take a calculus class.

Problem 6 *Prove the "backwards direction" of Theorem 1—that is, any decimal expression that is either finite or eventually repeating represents a rational number. (Hint: Can you rewrite the expression S from Problem 4 as a repeating decimal? Then generalize this and use the previous technique to solve the sums.)*

Solution: A finite expression terminating n places after the decimal is the same as those digits over 10^n (for example, $3.14 = 314/100$). For a repeating expression, we can write each repeating part this way, which forms a geometric series. For example,

$$0.\overline{365} = \frac{365}{1000} + \frac{365}{1000000} + \frac{365}{1000000000} + \cdots = \frac{365}{999}$$

by the geometric series formula.

Problem 7 *Prove the "forwards direction" of Theorem 1—that is, any rational number has a finite or eventually repeating decimal expansion. (Hint: Given any fraction, try rewriting it so that the denominator looks like the denominators you analyzed in the previous problem.)*

Solution: Given a rational number, we can find some integer $999\ldots9000\ldots0$ divisible by its denominator. (Many different ways to show this fact.) We can then rewrite this kind of denominator as a geometric series and then a repeating decimal. For example,

$$\frac{1}{14} = \frac{714285}{9999990} = 0.0\overline{714285}$$

2 Examples of Irrational Numbers

We now meet our first examples of numbers that we can show to be *irrational* (that is, a real number that isn't rational) using Theorem 1.

Problem 8 *Let α be the following decimal:*

$$\alpha = 0.101001000100001000001\ldots$$

(that is, every 1 is preceded by one more zero than the last) Show that α is irrational, by showing that this decimal expression is infinite and doesn't repeat.

Solution: α's decimal expression is infinite by definition because there are always more zeroes possible. Suppose it eventually repeated. The repeating part can't be all zero, since the expression contains 1's after any point, and any repeating 1's are some fixed distance apart (if the repeat starts with the i^{th} 1, the 1's are at most $2i-1$ apart). Since α's decimal expression eventually contains 1's further apart than that, it can't repeat.

The following examples will also be useful later. (Also note that we use the notation " $:=$ " meaning "is defined to be equal to".)

Definition 1 *Let b be an integer at least 2. The **base b Liouville constant** is given by*

$$L_b := \frac{1}{b^{1!}} + \frac{1}{b^{2!}} + \frac{1}{b^{3!}} + \frac{1}{b^{4!}} + \dots$$

Problem 9 *Show that the Liouville constant L_{10} is irrational.*

(Bonus) Show that all L_b are irrational. (Hint: Think about why Theorem 1 also works with any base b expression.)

Solution: The decimal expansion contains a 1 in every $k!$ spot with a 0 everywhere else. The proof that this is irrational is basically identical to Problem 8.

Of course, we don't need the decimal expansion to show that certain numbers are irrational. In fact, finding a (possibly infinite) decimal expansion for most numbers is impossible anyway. In the traditional example of an irrational number $\sqrt{2} = 1.4142\dots$, we could be stuck forever waiting for the decimal expansion to terminate or repeat, a possibility we can never rule out no matter how many digits we find.

Problem 10 *Show that $\sqrt{2}$ is irrational. (Hint: Any fraction can be fully reduced—that is, the numerator and denominator made to be coprime. So assume $\sqrt{2} = a/b$ is a fully reduced fraction. You should derive some contradiction.)*

Solution: Suppose $\sqrt{2} = a/b$ is fully reduced. In particular, this means a and b can't both be even. Squaring both sides of the previous equation gives $a^2 = 2b^2$, so a is even. But then we can write $a = 2c$, so $4c^2 = 2b^2 \Rightarrow b^2 = 2c^2$, so b is also even, which is a contradiction.

Problem 11 *Can you modify your solution to the previous problem to show that $\sqrt{3}$ is irrational? How about $\sqrt[3]{2}$? How about $\sqrt{2} + \sqrt{3}$?*

Solution: The proofs for $\sqrt{3}$ and $\sqrt[3]{2}$ are similar. $\sqrt{2} + \sqrt{3}$ has a different proof. Suppose it were rational. Then $1/(\sqrt{2} + \sqrt{3}) = \sqrt{3} - \sqrt{2}$ would also be rational, so $(\sqrt{2} + \sqrt{3}) + (\sqrt{3} - \sqrt{2})$ would also have to be rational. But that equals $2\sqrt{3}$, which contradicts the fact that $\sqrt{3}$ is irrational.

Finally, we have the two most famous examples of irrational numbers, e and π, though we will not prove the irrationality of π. To prove the irrationality of e, we use the following definition (which might look different from the usual decimal expansion $e = 2.71828\ldots$, but that one can check is the same using calculus and a calculator):

Problem 12 *Show that the number given by*

$$e := \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

is irrational. (Hint: Use a combination of the two techniques we've learned—first assume $e = a/b$, then look at all terms after $1/b!$.)

Solution: Suppose $e = a/b$ where a and b are integers. Then $(b-1)!a = b!e$ is an integer. We have

$$b!e = \left(b! + \frac{b!}{1!} + \frac{b!}{2!} + \cdots + b + 1\right) + \left(\frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \ldots\right)$$

The first parentheses contain an integer, so the second parentheses must contain an integer as well. But

$$\begin{aligned} 0 < \frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \ldots &= \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \ldots \\ &\leq \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)^2} + \ldots \\ &= \frac{1}{b+1}\frac{b+2}{b+1} < 1 \end{aligned}$$

so we have an integer greater than zero and less than one, a contradiction.

3 Continued Fraction Approximations

Definition 2 *A **continued fraction** is a (possibly infinite) expression of the form*

$$[a_0; a_1, a_2, a_3, \ldots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Problem 13 *Rewrite the following continued fractions as normal fractions (continued on next page).*

- $[1; 2]$

  Solution: $3/2$

- $[0; 1, 5]$

  Solution: 5/6

- $[3; 7, 16]$

  Solution: 355/113

Problem 14 *Rewrite the following rational numbers as continued fractions.*

- $\frac{6}{7}$

  Solution: $[0; 1, 6]$

- $\frac{5}{3}$

  Solution: $[1; 1, 2]$

- $\frac{17}{12}$

  Solution: $[1; 2, 2, 2]$

Problem 15 *Show that a continued fraction represents a rational number if and only if it's finite. (Hint: The backwards direction is quick. For the forwards direction, give a way to turn any rational number into a continued fraction, like in the previous problem.)*

Solution: ($\Leftarrow$) follows from the fact that we only use integers and fractions. ($\Rightarrow$): Given any rational number, we separate its integer part, take the reciprocal of its fractional part, and repeat (see full algorithm on next page).

In general, the same algorithm as you have found in the previous problem finds the continued fraction expression of any real number. To summarize,

1. Take the integer part of the number, which will be a_0. Subtract a_0 from the number.
2. Take the reciprocal of the resulting number.
3. The integer part of that reciprocal is a_1, which we then subtract in a repeat of Step 1. Then repeat Step 2, and so on.

Of course, by Problem 15 we should expect this algorithm to run infinitely if we start with an irrational number. But decimal expansions could be infinite as well, and we could still describe them if they repeated (say).

Problem 16 *Using a calculator, find the first eight terms of the continued fraction expressions of the following real numbers. Do you see a pattern (not necessarily just a repeat) in any of them?*

- $\sqrt{2}$

  Solution: $[1; 2, 2, 2, 2, 2, 2, 2]$

- $\sqrt{3}$

  Solution: $[1; 1, 2, 1, 2, 1, 2, 1]$

- $\sqrt[3]{2}$

  Solution: $[1; 3, 1, 5, 1, 1, 4, 1]$

- $\sqrt{2} + \sqrt{3}$

  Solution: $[3; 6, 1, 5, 7, 1, 1, 4]$

- e

  Solution: $[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \dots]$ *(Note: Pattern is not obvious without more terms.)*

- π

  Solution: $[3; 7, 15, 1, 292, 1, 1, 1]$

- L_{10} *(Bonus: Think about what L_b's continued fraction looks like in general.)*

  Solution: $[0; 9, 11, 99, 1, 10, 9, 999999999999]$

Of course, since we can't write down infinitely long expressions, all the infinite expressions we've written down so far are estimates. For example, we can estimate π by writing down the first three (3.14), four (3.141), or five (3.1415) digits in its decimal expansion. Some of these estimates are "better" than the others, and one way we can measure this is by finding the error.

Definition 3 *Given a real number x and an estimate y for x, the* ***absolute error*** *of y is $|x - y|$.*

Problem 17 *Give the (absolute) error for the following decimal estimates of $\sqrt{2}$ in terms of $\sqrt{2}$:* $1.4, 1.41, 1.414$.

Solution: $\sqrt{2} - 1.4, \sqrt{2} - 1.41, \sqrt{2} - 1.414$

Of course, we can't write down $\pi - 3.1415$ any more than we can write down π itself, but as long as the error of an estimate is small enough, its exact value often doesn't matter.

Problem 18 *Show that the (absolute) error in the "take the first n digits of the decimal expansion" estimate (for any real number) is at most 10^{1-n}.*

Solution: Any difference of more than 10^{1-n} changes the $1 - n^{th}$ digit after the decimal.

Put differently, in the previous problem we showed that for the decimal approximation,

$$\text{Error} \leq \frac{1}{\text{Denominator}}$$

since the decimal approximation is essentially the same as writing the approximation as a fraction over 10^{n-1}. For instance, $3.14 = 314/100$, and our error bound is about $1/100$. However, using continued fractions, we can do much better.

Problem 19 *Check that the first four estimates given by the "take the first n parts of the continued fraction expansion" estimate for $\sqrt{2}$ satisfy*

$$Error \leq \frac{1}{Denominator^2}$$

(These estimates are the continued fractions $[1], [1; 2], [1; 2, 2], [1; 2, 2, 2]$. I encourage you to do this without using the decimal expansion or a calculator, though it's good to check your answers with those.)

Solution: We start with the estimate $\sqrt{2} - 1 < 1/2 < 1^2$. Squaring both sides gives the estimate $3 - 2\sqrt{2} < 1/4 \Rightarrow 3/2 - \sqrt{2} < 1/8 < 2^2$. Squaring both sides gives the estimate $17/12 - \sqrt{2} < 1/192 < 12^2$. Similar tricks work for $7 - \sqrt{5}$.

Problem 20 *Compute the first few continued fraction expansions for π with a calculator. Which ones give very small errors (compared to, say, their denominators squared)?*

Problem 21 *In general, where should you cut off a continued fraction to get the best estimate (in terms of error)?*

Solution: Right before a really big number.

Problem 22 *Find a continued fraction estimate for L_{10} that you think will give very small error. Try estimating that error (or calculate it, if your calculator has enough significant digits).*

Solution: $[0; 9, 11]$ is good enough, and $[0; 9, 11, 99, 1, 10, 9]$ is extremely close.

4 Repeating Continued Fractions and Quadratics

A couple of the continued fraction expansions we computed previously did repeat, much like a rational number's decimal expansion. For these continuous functions, we can solve for their exact value.

Problem 23 *Find the value of* $[1; 1, 1, 1, \ldots]$. *(Hint: Let*

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}$$

Find another x *in this expression.)*

Solution: We see that

$$x = 1 + \frac{1}{1 + x}$$

so that $x(x+1) = (x+1) + 1$, and by the quadratic formula $x = (1 + \sqrt{5})/2$.

Problem 24 *Earlier, we calculated* $\sqrt{2} = [1; 2, 2, 2, \ldots]$ *and* $\sqrt{3} = [1; 1, 2, 1, 2, \ldots]$ *experimentally. Prove these by showing that* $[2; 2, 2, 2, \ldots] = \sqrt{2} + 1$ *and* $[0; 1, 2, 1, 2, \ldots] = \sqrt{3} - 1$*, respectively.*

Solution: Similar to the previous problem.

We see that repeating continued fraction expressions give quadratic equations, so that in particular we can express numbers like square roots with repeating continued fractions. We give numbers like this a special name.

Definition 4 *A real number is* ***quadratic*** *if it is the root of a quadratic polynomial with rational coefficients.*

Problem 25 *Prove that a real number has a repeating continued fraction if and only if it is quadratic and irrational.*

We appear to have found some numbers that are not quadratic, but like with decimal expansions we run into the difficulty of proving that an expression is infinite and non-repeating. We will not prove today that, for example, $\sqrt[3]{2}$ and e are not quadratic, but there is one example that is simple enough.

Problem 26 *Show that the Liouville constants* L_b *are not quadratic. (Hint: Describe the base* b *expansions for* L_b *and* L_b^2*. When you plug them into the generic quadratic* $ax^2 + bx + c$*, why can you not get zero?)*