Factorization

ORMC

03/31/24

1 GCD Review

The greatest common divisor of two integers m and n, $\gcd(m,n)$, is just that, the greatest positive integer that divides both of them.

Problem 1.1. Show that $\gcd(m, m+n) = \gcd(m,n)$ and $\gcd(m, n-m) = \gcd(m,n)$.

Problem 1.2. If $m < n$, what possibilities are there for $\gcd(m+n, n-m)$?

Fact 1.3. *If m, n, c are integers, then there are integers a, b such that $am + bn = c$ if and only if $\gcd(m,n)$ divides c.*

2 Problems from *Putnam and Beyond*

Problem 2.1. Show that for positive integers a and b, the product $(36a+b)(a+36b)$ cannot be a power of 2.

Problem 2.2. Prove that $1! \cdot 2! \cdot \cdots \cdot 100!$ is not a perfect square, but there is some $1 \le n \le 100$ such that the product skipping $n!$, that is, $1! \cdot 2! \cdot \cdots \cdot (n-1)! \cdot (n+1)! \cdot \cdots \cdot 100!$, is a perfect square.

Problem 2.3. Let n, a, b be positive integers. Prove that

$$\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1.$$

3 Problems from *Problem Solving Through Problems*

Problem 3.1. Find the smallest positive integer a for which

$$1001x + 770y = 1,000,000 + a$$

is possible, and show that it has then 100 positive integer solutions.

Problem 3.2. Prove that any two successive Fibonacci numbers F_n,F_{n+1} with $n > 2$ are relatively prime.

4 Competition Problems

Problem 4.1 (2009 BAMO Problem 5)**.** A set S of positive integers is called magic if for any two distinct members of S, i and j,

$$\frac{i+j}{\gcd(i,j)}$$

is also a member of S. Find and describe all finite magic sets.

Problem 4.2 (2008 USAMO Problem 1)**.** Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1 \dots, k_n$, all strictly greater than 1, such that $k_0 k_1 \cdots k_n - 1$ is the product of two consecutive integers.

Hint: Induction on n. If $k_0 k_1 \dots k_n - 1$ is the product of two consecutive integers, find some $k_{n+1} > 1$ such that $k_0 k_1 \dots k_{n+1} - 1$ is the product of two consecutive integers, with k_{n+1} relatively prime to $k_0, \dots, k_n$.

Problem 4.3 (2008 USAMO Problem 5)**.** Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1 r_1 + a_2 r_2 + a_3 r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \le y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

Hint: Along with each triple (r_1, r_2, r_3), write a triple (a_1, a_2, a_3) to go with it. Track how the as change with the rs, and try to shrink $a_1^2 + a_2^2 + a_3^2$.

Problem 4.4 (2009 USAMO Problem 6)**.** Let $s_1, s_2, s_3, \dots$ be an infinite, nonconstant sequence of rational numbers, meaning it is not the case that $s_1 = s_2 = s_3 = \cdots$. Suppose that $t_1, t_2, t_3, \dots$ is also an infinite, nonconstant sequence of rational numbers with the property that $(s_i - s_j)(t_i - t_j)$ is an integer for all i and j. Prove that there exists a rational number r such that $(s_i - s_j)r$ and $(t_i - t_j)/r$ are integers for all i and j.

Problem 4.5 (2018 Putnam A1)**.** Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Hint: Turn this into an integer equation, and consider both factorization and arithmetic mod 3.

Problem 4.6 (2015 Putnam A5)**.** Let q be an odd positive integer, and let N_q denote the number of integers a such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that N_q is odd if and only if q is of the form p^k with k a positive integer and p a prime congruent to 5 or 7 modulo 8.