Worksheet:

Throughout this worksheet all rings have 0 \neq 1.

A ring homomorphism \( f \) is a function between two rings \( f : R \to S \), satisfying the following properties:

- **addition preserving:** For all \( a, b \) in \( R \)
  \[ f(a + b) = f(a) + f(b) \]

- **multiplication preserving:** For all \( a, b \) in \( R \)
  \[ f(a \cdot b) = f(a) \cdot f(b) \]

- **unit preserving:**
  \[ f(1_R) = 1_S \]

**Problem 9.1:** Show that the following functions are ring homomorphisms:

- \( f : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \), that sends an integer to its residue modulo 4.
- \( g : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \), that sends a number to its residue modulo 2.
- \( h : \mathbb{R} \to \mathbb{C} \), defined by \( h(a) = a \) for all \( a \) in \( \mathbb{R} \).

**Solution 9.1:**
Problem 9.2: Show that for any ring homomorphism $f : R \to S$, we have that

$$f(0_R) = 0_S$$

Solution 9.2:
Problem 9.3: Show that a ring homomorphism $f : R \to S$ is one-to-one if and only if the preimage of $0_S$ is only $0_R$, i.e. $f$ is one-to-one if and only if $f(x) = 0_S$ implies $x = 0_R$.

Show that if $f : R \to S$ is a ring homomorphism and $R$ is a field, then $f$ is one-to-one.

Is this true when only $S$ is a field?

Solution 9.3:
If a ring homomorphism $f : R \to S$ is furthermore a bijection, then we say $f$ is an isomorphism.

**Problem 9.4:** Let $f : R \to S$ be an isomorphism, show that if $R$ or $S$ is a field, then both of them are fields.

Is this true for ring homomorphisms that are not isomorphisms?

**Solution 9.4:**
Problem 9.5: Show that the composition of two ring homomorphisms is a ring homomorphism.
Solution 9.5:
Let \( \mathbb{F} \) be a finite field.

**Problem 9.6:** Show that there are no ring homomorphisms \( f : \mathbb{R} \to \mathbb{F} \).

**Solution 9.6:**
Problem 9.7: Give sufficient and necessary conditions on $m$ and $n$ for the existence of a ring homomorphism:

$$f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}.$$ 

How many such homomorphisms exist for each possible $m, n$?

Solution 9.7:
Problem 9.8: Describe all ring homomorphisms $f : \mathbb{Z}[x] \rightarrow \mathbb{Z}[x]$.

Solution 9.8:
Let $p$ be a prime number.

**Problem 9.9:** How many ring homomorphisms of the form $f : \mathbb{F}_p \to \mathbb{F}_q$ exist?

**Solution 9.9:**
Problem 9.10: How many ring homomorphisms of the form $f: \mathbb{F}_4 \to \mathbb{F}_4$ are there?
How many ring homomorphisms of the form $f: \mathbb{F}_9 \to \mathbb{F}_9$ are there?

Solution 9.10: