OLGA RADKO MATH CIRCLE, WINTER 2024: ADVANCED 3

FERNANDO FIGUEROA AND JOAQUÍN MORAGA

Worksheet :

Throughout this worksheet all rings have $0 \neq 1$.

A ring homomorphism f is a function between two rings $f : R \to S$, satisfying the following properties:

- addition preserving: For all a, b in R
$$f(a + b) = f(a) + f(b)$$
- multiplication preserving: For all a, b in R
$$f(a \cdot b) = f(a) \cdot f(b)$$
- unit preserving:
$$f(1_R) = 1_S$$

Problem 9.1: Show that the following functions are ring homomorphisms:

- $f : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$, that sends an integer to its residue modulo 4.
- $g : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, that sends a number to its residue modulo 2.
- $h : \mathbb{R} \to \mathbb{C}$, defined by $h(a) = a$ for all a in $\mathbb{R}$.

Solution 9.1:

Problem 9.2: Show that for any ring homomorphism $f : R \to S$, we have that

$$f(0_R) = 0_S$$

Solution 9.2:

Problem 9.3: Show that a ring homomorphism $f : R \to S$ is one-to-one if and only if the preimage of 0_S is only 0_R, i.e. f is one-to-one if and only $f(x) = 0_S$ implies $x = 0_R$.

Show that if $f : R \to S$ is a ring homomorphism and R is a field, then f is one-to-one.

Is this true when only S is a field?

Solution 9.3:

If a ring homomorphism $f : R \to S$ is furthermore a bijection, then we say f is an isomorphism.

Problem 9.4: Let $f : R \to S$ be an isomorphism, show that if R or S is a field, then both of them are fields.

Is this true for ring homomorphisms that are not isomorphisms?

Solution 9.4:

Problem 9.5: Show that the composition of two ring homomorphisms is a ring homomorphism.
Solution 9.5:

Let $\mathbb{F}$ be a finite field.

Problem 9.6: Show that there are no ring homomorphisms $f : \mathbb{R} \to \mathbb{F}$.

Solution 9.6:

Problem 9.7: Give sufficient and necessary conditions on m and n for the existence of a ring homomorphism:

$$f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}.$$

How many such homomorphisms exist for each possible m, n?

Solution 9.7:

Problem 9.8: Describe all ring homomorphisms $f : \mathbb{Z}[x] \to \mathbb{Z}[x]$.
Solution 9.8:

Let p be a prime number.

Problem 9.9: How many ring homomorphisms of the form $f : \mathbb{F}_p \to \mathbb{F}_q$ exist?

Solution 9.9:

Problem 9.10: How many ring homomorphisms of the form $f : \mathbb{F}_4 \to \mathbb{F}_4$ are there?

How many ring homomorphisms of the form $f : \mathbb{F}_9 \to \mathbb{F}_9$ are there?

Solution 9.10:

UCLA Mathematics Department, Los Angeles, CA 90095-1555, USA.
Email address: fzamora@math.princeton.edu

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA.
Email address: jmoraga@math.ucla.edu