WINTER REVIEW

MAX STEINBERG FOR OLGA RADKO MATH CIRCLE
ADVANCED 2

1. Introduction

Hi everyone! In today's packet, we will solve some fun problems that use some of the topics we learned earlier in the quarter! Some problems might be very similar to ones you've seen before, so it may help to look back. If you don't remember a particular topic, feel free to reread parts of the earlier packets or ask an instructor. You can do the problems in any order, but they are rated by difficulty.

2. Lambda Calculus ★

Binary Lambda Calculus is a way to encode lambda calculus on a computer. Letters take up a lot of space, so instead of letters, we will use *numbers* for variables:

$$\lambda x.\lambda y.xy \Rightarrow \lambda\lambda.12$$

Recall that we can combine our lambdas into a single lambda $\lambda\lambda xy.xy$, and then we rewrite it to remove any letters: $\lambda\lambda.??$. We replace x with 1 since it is the first variable, and y with 2 since it is the second variable, so we get $\lambda\lambda.12$.

Now, how do we convert this into binary? Very simple. We replace every λ with 00, every variable (given as a number) with that number of 1s followed by a 0 (so 1 becomes 10, 2 becomes 110, 3 becomes 1110, etc.), and every application is written as $01xy$, where x and y can be either lambdas or variables. So our lambda becomes

$$\lambda\lambda.12 \Rightarrow 00\,00\,0110110 = 00000110110$$

$$\underbrace{00}_{\lambda}\underbrace{00}_{\lambda}\underbrace{01}_{\textit{apply}}\underbrace{10}_{1}\underbrace{110}_{\text{to }2}$$

To parse 00000110110, we go piece by piece. We have a 00, so we write "λ". We have another 00, so we write another "λ". We have a 01, so we have an application. The first argument is 10, which is variable 1. The second argument is 110, which is variable 2. So we get $\lambda\lambda.12$.

Problem 1. Convince yourself that this is an unambigous encoding.

Problem 2. Convert our boolean functions to binary lambda calculus (recall $T = \lambda x.\lambda y.x$ and $F = \lambda x.\lambda y.y$).

Problem 3. Convert the following binary lambda calculus functions to usual lambda calculus.

(1) 0010
(2) 000010
(3) 0000000101101110011101110

3. Determinants ★★

You may use all of the results of determinants that you proved in the Determinants Packet.

Problem 4. Let A be an invertible n-by-n matrix. Prove that $\det(A) \neq 0$ and that $\det(A^{-1}) = \frac{1}{\det(A)}$.

Problem 5. Let M be a 3-by-3 matrix with $\det(M) = 1$ and $M \cdot M^t = I$. Prove that $\det(M - I) = 0$.

Problem 6. Let A and B be two n-by-n matrices, where A is invertible. Prove that $\det(ABA^{-1}) = \det(B)$.

Problem 7. The numbers 23028, 31882, 86469, 6327, and 61902 are all divisible by 19. Show that the determinant of the matrix

$$\begin{bmatrix} 2 & 3 & 0 & 2 & 8 \\ 3 & 1 & 8 & 8 & 2 \\ 8 & 6 & 4 & 6 & 9 \\ 0 & 6 & 3 & 2 & 7 \\ 6 & 1 & 9 & 0 & 2 \end{bmatrix}$$

is divisible by 19.

4. Proofs ★★★

Problem 8. Recall that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} =: T_n$$

are the triangular numbers. Let TT_n be the n-th *tetrahedral* number, given by

$$\sum_{i=1}^{n} T_i =: TT_n$$

Prove that

$$TT_n = \frac{n(n+1)(n+2)}{6}$$

Problem 9. Prove that there is a nonzero digit i ($i \in \{1, 2, \ldots, 9\}$) that occurs infinitely many times in the base-10 expansion of π. (The claim that this is true for *all* digits is an open problem!)

Problem 10 (also a problem in the Proofs packet)**.** What is the probability that two randomly chosen positive integers are relatively prime?

5. Groups ★★★★

Problem 11. Let p be a prime number and G a group with $|G| = p$. Prove that G is abelian (commutative).

Problem 12. Find all groups of order 4.

Problem 13. Prove that every finite group is a subgroup of a symmetric group.

6. Arithmetic Functions ★★★★★

Recall that a positive integer n is said to be *perfect* if $\sigma(n) = 2n$ (where $\sigma(n)$ is the sum of the divisors of n).

Problem 14. Prove the **Euclid-Euler Theorem**: every even perfect number is of the form $2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime.

Recall the definition of a Dirichlet series:

Definition 1 (Dirichlet series)**.** Given an arithmetic function $f : \mathbb{N} \to \mathbb{C}$ (note that we now work over $\mathbb{C}$ rather than $\mathbb{R}$), we can define a **Dirichlet series** associated to f:

$$\mathcal{D}_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

This can be treated as a function $\mathcal{D}_f(s) : \mathbb{C} \to \mathbb{C}$.

The most famous example of a Dirichlet series is the **Riemann zeta function**,

$$\zeta(s) = \mathcal{D}_{\mathbf{1}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Problem 15 (also a problem in the Arithmetic Functions packet)**.** Prove that

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{(f * g)(n)}{n^s}$$

for any $s \in \mathbb{C}$ where all sums converge.

Problem 16 (also a problem in the Arithmetic Functions packet)**.** Prove that

$$\mathcal{D}_\mu(s) = \frac{1}{\zeta(s)}.$$

Problem 17 (also a problem in the Arithmetic Functions packet)**.** Prove that

$$\frac{\zeta(s-1)}{\zeta(s)} = \mathcal{D}_\phi(s)$$

Problem 18. Prove that

$$\zeta(s-1)\zeta(s) = \mathcal{D}_\sigma(s)$$