OPTIONS IN FINANCE

OLGA RADKO MATH CIRCLE
ADVANCED 2

1. Introduction

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Ever heard of wall street bets? How about dogecoin? Gamestop? I haven't. In this worksheet, we will introduce option contracts and prove pricing bounds and properties under some reasonable assumptions. An **asset** in finance is a resource that can be owned and has some economic value. In other words, you could sell it for cash! One type of asset we will focus on this week is a **stock**. A share of stock in a company represents ownership of that fraction of the company. Owning shares of a stock is therefore an asset; the company is (hopefully) worth some money and owning a fraction of it has economic value. [1]

Assets can be even less direct than stocks. For example, imagine you enter a contract in which someone must sell you stock at some fixed price on a later date. This contract is an asset, it has economic value. Theoretically, you can sell this contract to a third party and they will take your place in the agreement. We will explore assets of this type, called **derivatives**. [2]

2. Options

Definition 1 (Call option)**.** A (European) **call option** with strike price K and maturity T is a contract between two parties giving one party the option to buy the underlying stock for price K at an agreed expiration time T.

Buying the option is called *long call*, while selling the option is a *short call*.

Let's explore the circumstances under which we could make (or lose) money. The first thing to note is that we would only choose to buy the asset through the option if it were cheaper than in the market. In other words, we would only **exercise** the option if $K < S_T$. If we were to do this, then we could buy the asset for K and sell it to the market for S_T, giving us a payoff of $S_T - K$. On the other hand, if $K \geq S_T$ then we would not exercise the option. The payoff in this scenario would simply be 0. Therefore, we can describe our possible earnings by the piecewise function

$$(S_T - K)^\dagger = \max(S_T - K, 0) = \begin{cases} 0 & S_T \leq K \\ S_T - K & S_T \geq K \end{cases}$$

The dagger in the notation is used to indicate the max between the function and 0, $(f)^\dagger = \max(f, 0)$.

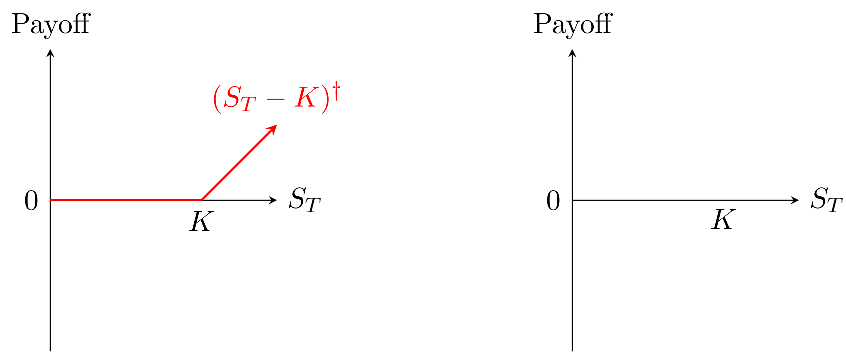
Problem 1. Philliam Luge wants to owns 300 call contracts for American Airlines (NASDAQ: AAL) because it's soon to be summer and if there's one sure thing about human nature, it's that they love going where they aren't. Travel is sure to pick back up! People can't stay in one place for too long. The strike price for the contracts is \$23 and the current price is \$20. How much money will Philliam make if, no, WHEN the share price increases to \$35 at the expiration dates of the contracts.

[1] Nothing in this worksheet constitutes professional and/or financial advice, nor does any information in the worksheet constitute a comprehensive or complete statement of the matters discussed or the law relating thereto. Math Circle is not a fiduciary by virtue of any person's use of or access to the worksheet or content.

[2] By writing your name on this worksheet you hereby relinquish your right to receive Social Security funding upon retirement at the ripe age of 68.

Definition 2. The **payoff function** of a portfolio of assets is the value of the portfolio at a given maturity time T. Effectively, this is the amount of money you would make or lose if you sold everything at market value. This does not include any costs or money earned in setting up the portfolio.

We can plot the payoff of a long call with strike price K and maturity T as a function of the stock price S_T at maturity.



Problem 2. Find and plot the payoff function for a short call with strike price K. Write the payoff function in dagger notation. (Remember that as the seller of the option, you do not decide whether to exercise.)

Definition 3 (Put option)**.** A (European) **put option** with strike price K and maturity T is a contract between two parties giving one party the option to sell the underlying stock for price K and maturity time T.

Buying the option is called *long put*, while selling the option is a *short put*.

Problem 3. Nae Serre doesn't believe in Tesla (NASDAQ: TSLA). I mean c'mon, the fundamentals just aren't there! They are way overvalued...Are you telling me they are worth more than all other car companies combined?! They've only sold like a million cars! Eventually the share price will come back down to Earth, unlike their cars, which are being sent up into space. Anywho, Nae bought 2000 Tesla (NASDAQ: TSLA) put contracts with a strike price of \$500 for super cheap since the current stock price is \$952. Explain why the contracts were practically free and then compute Nae's profit when Tesla crashes to a still overvalued \$250.

Remark 1. Long and short refer to the asset itself, not necessarily the underlying stock. For this reason, being long on a put contract means you believe the **value of the contract** will go up.

Problem 4. If your position is a short put, are you hoping that the stock price goes up or down?

Problem 5. Find and plot the payoff function for:

(a) a long put with strike price K
(b) a short put with strike price K
(c) a share of stock

Problem 6. Suppose the price of share of GameStop (GME) is \$150 and a call option with strike price \$150 expiring in one week costs \$5. You found \$1500 in an old The Chronicles of Narnia book and decide to invest it. Compare the payoffs of the following two scenarios:

(1) Spending all your money on shares of GME
(2) Spending all your money on call options

What are the pros and cons of each strategy?
What is the profit of each if the price for GME goes up to \$300? What about \$152? \$148?

3. Trading strategies

Definition 4. A **trading strategy** is a fixed plan to buy or sell assets (stocks, goods, contracts, etc).

Definition 5. A **covered call** is a trading strategy in which you buy the stock and sell a call option at time $t = 0$. At maturity T of the call option, you sell the stock. Specifically, if the call is exercised, you sell it for K. If the call is not exercised, you sell it in the market for S_T.

Problem 7. Find the payoff of a covered call. The payoff does not take into account the cost of setting up a trading strategy (in this case, buying the stock and selling the option). Why might one choose a covered call instead of a short call?

Definition 6. The **net profit** of a trading strategy is the sum of the payoff and the set-up cost.

Example 1. If a call with strike price K and maturity T costs $C(K,T)$, then the net profit of a long call is $(S_T - K)^\dagger - C(K,T)$.

Remark 2. The set-up cost is paid to the brokerage company for providing a service to the investor.

Problem 8. Find the net profit function of a short put with price $P(K,T)$ as a function of the stock price at maturity, S_T.

Problem 9. Find the net profit of a covered call as a function of S_T.

Definition 7. Buying a share of stock for price S_0 at the start and selling at the end gives a net profit of $S_T - S_0$. We call this being **long** on a stock. Similarly, one can sell the share at the start for price S_0 and buy it at the end for S_T. This is called **short selling**, since you are speculating that the stock will go down in value.

Problem 10. Lick Lebesgue shorted 100 shares of Lyft (NASDAQ: LYFT) when it was \$31. Lyft (NASDAQ: LYFT) released it's Q1 earnings on May 4th, 2022 and instantly tanked to \$20. If Lick closes out their position (buys back the shares), how much money did they make?

Problem 11. Find a trading strategy with stock and a call option that simulates a long put option with strike price S_0. A trading strategy is said to simulate another if their net profits are equal.

Problem 12. A **straddle** is a trading strategy consisting of a long call and a long put with the same strike price and maturity. Find and plot the payoff function. What are you hoping happens to the stock price for a straddle?

Problem 13. Pip Bellevue III of Saint Oculus doesn't know what's going to happen, but she knows something will happen. She's almost certain that Apple (NASDAQ: AAPL), stock will tank but she's afraid that people might finally be buying phones again and will eat up the iPhone 37 Pro. Naturally, she took a straddle with strike price \$323 when they announced the high-tech iPhone 37 Pro. The straddle just expired and the price now is \$223. How much money did Pip Bellevue III of Saint Oculus earn?

Problem 14. Elmer Clayborne is extremely excited about virtual reality. In an email to his broker Phil Smell, Elmer requests a "meta option." Phil is a squid. What does Phil purchase with Elmer's hard-earned money?

4. Arbitrage, No Arbitrage, and applications

Definition 8. **Arbitrage** is a trading strategy in which the net profit is always non-negative and somewhere positive. In other words, it's a risk-free way either make money or come out even.

Definition 9 (**No Arbitrage**)**.** In an *ideal* market (a technical term we will not define here), there should be **no arbitrage**. In other words, there is no such thing as free money. In *other* other words, you've gotta risk it for the biscuit.

One way to think about this is that risk should be viewed as a cost. In order to come out even, there must be risk involved in order to have a positive payoff. It's often the case that the bigger the risk, the larger the possible payoff.

Problem 15. Show that the prices of calls and puts must both be positive. In other words, the party with the option to exercise is the one paying for the contract.

Definition 10. Let $K_1 < K_2$. A (long) **butterfly spread** is a trading strategy consisting of a long call with strike K_1, a long call with strike K_2, and two short calls with strike $\frac{K_1+K_2}{2}$.

Problem 16. Pluto van Plutt[3] knows that Coca-Cola (NYSE: KO)[4] is the most stable stock in history. People always have and always will drink Coke. Nobody's drinking more of it, and nobody's drinking less of it. Don't like Coke? Try Coke Zero! Down on yourself? Try Coke Hero! Naturally, Pluto buys 20 contracts of a butterfly with strike prices $\$60, \$65, \$70$. Come expiration time, Coca-Cola (NYSE: KO) will inevitably remain at its price of \$65 per share. How much money will Pluto make?

Problem 17. Find and plot the payoff function for a butterfly spread. What are you hoping happens to the stock price?

Definition 11. A function $f : \mathbb{R} \to \mathbb{R}$ is **convex** if the line segment between any two points on the graph does not lie below the graph.

If f is continuous, this is equivalent to the condition that $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$ for all x, y.

Problem 18. Use the principle of no arbitrage and the payoff of the butterfly spread to prove that call option pricing is convex with respect to strike price. It is a reasonable assumption that pricing is a continuous function of strike price.

Problem 19. Construct a trading strategy and prove a similar statement for put pricing.

Problem 20. Show that the price of a call option, $C(K, T)$, must be less than the initial price of the stock, S_0. What arbitrage would be available if this weren't the case?

Problem 21. Prove that the price of a put, $P(K, T)$, is less than the strike price K. What arbitrage is available if $K < P(K, T)$?

One strategy we often employ to develop bounds on options pricing is to construct two portfolios (trading strategies) which result in the same payoff function. Then our no arbitrage principle implies that the set up costs must be the same too! Otherwise, you could buy the cheaper portfolio and sell the more expensive one. You will profit from the set-up cost difference and come out even from the cancelling payoffs, giving you positive net profit overall.

5. Interest rates and the Put-Call parity

In this section, we introduce interest rates and use them to prove inequalities of option pricing under certain assumption.

Definition 12. An **interest rate** is the proportion of a loan that is charged as interest to the borrower.

In our case, we will consider continuously compounded interest rates. The rates will be given per annum (so we measure time in years) and we will assume that money can be both borrowed and invested with this interest rate.

Specifically, this means that an initial amount of money, K, invested in a savings account with interest rate r will grow to Ke^{rt} after t years. Similarly, if you borrow $\$K$, you must repay Ke^{rt} after t years.

Problem 22. Show that $C(K, T) \geq S_0 - Ke^{-rT}$. Set up the following two portfolios. Portfolio A is a long call and $\$Ke^{-rT}$ invested at interest rate r. Portfolio B is the stock.

(a) What is the payoff of portfolio A?
(b) What is the set up cost of portfolio A? Remember that you start with nothing, so any money you use is part of the setup cost.
(c) What is the payoff of portfolio B?
(d) What is the setup cost? (How much did you pay for the stock?)
(e) Compare their payoffs and conclude that the setup costs must have the same relation.

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[3]There is some joke about Coca-Cola to Pluto here especially now that Pluto isn't much more significant than the moon astronomically. I'll workshop this and get back to you.

[4]If the abbreviation KO isn't an indication that this is a good idea, I don't know what is.

Combining this with our earlier upper bound for $C(K,T)$, we get $S_0 - Ke^{-rT} \leq C(K,T) \leq S_0$

Problem 23. Come up with a similar bound for the price of a put option.

Theorem (Put-call parity)**.** Let $C(K,T)$ and $P(K,T)$ be the price of a call and put, respectively. Let S_0 be the price at time 0 and r be the interest rate. Then

$$C(K,T) + Ke^{-rT} = S_0 + P(K,T)$$

Problem 24. Eilon bought 500 call contracts for Advanced Micro Devices (NASDAQ: AMD) at a price of \$11.25 per contract when AMD was at \$90. The strike price was \$100 and maturity time one year. The current interest rate is 3%. How much would a put cost with the same strike price and maturity? [5]

Problem 25. We will prove the theorem in steps:

(a) Let portfolio A consist of a long call and $\$Ke^{-rT}$ invested in a savings account. What is the payoff of this portfolio? What is the setup cost?

(b) Let portfolio B consist of a long put and the stock. What is the payoff of this portfolio? What is the setup cost?

(c) Compare the two portfolios and prove the theorem!

Problem 26. Improve the upper bound for put prices by considering interest rates.

[5] He bought this when AMD was \$90, right before they released STELLAR earnings. The next day AMD jumped to \$99, making each call worth \$16.85. Any smart man would sell the call and make a profit of 50% in only one day. But not only does Eilon believe in the company, he also knows he'll regret not selling the contracts. Oh wait...